

## Math 2080: Differential Equations

### Worksheet 7.4: The wave equation

NAME:

Let  $u(x, t)$  be defined for  $0 \leq x \leq 1$  and  $t > 0$ , and consider the following PDE

$$u_{tt} + 2\beta u_t = u_{xx}, \quad u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = x(\pi - x), \quad u_t(x, 0) = 1.$$

where  $0 < \beta < 1$  is a constant. This is the wave equation where the  $u_t$  term models transverse vibrations being in a medium that imparts a resistance proportional to the instantaneous velocity.

(a) Describe and sketch this situation at  $t = 0$ .

(b) Assume that there is a solution of the form  $u(x, t) = f(x)g(t)$ . Plug this back into the PDE and get an ODE for  $g(t)$  and a BVP for  $f(x)$ .

(c) The BVP should be familiar:  $f'' = -\lambda f$ ,  $f(0) = f(\pi) = 0$ , and we've seen that  $\lambda_n = n^2$  for  $n = 0, 1, 2, \dots$  and  $f_n(x) = b_n \sin nx$ . Solve the ODE for  $g(t)$ .

(d) Write down the general solution to this PDE.

(e) Use the initial conditions to find the particular solution solving the boundary and initial conditions.

(f) What is the long-term behavior of this system? Give both a mathematical and physical justification.