

Math 2080: Differential Equations

Worksheet 7.8: The two-dimensional wave equation

NAME:

1. Consider the following initial/boundary value problem for the 2D wave equation.

$$\begin{aligned}u_{tt} = c^2 \nabla^2 u, \quad & u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\ & u(x, y, 0) = x(\pi - x)y(\pi - y), \quad u_t(x, y, 0) = 1.\end{aligned}$$

- (a) Briefly describe a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial conditions. Sketch the initial displacement, $u(x, y, 0)$.

- (b) Assume that the solution has the form $u(x, y, t) = f(x, y)g(t)$. Find u_{xx} , u_{yy} , u_t , and u_{tt} .

- (c) Plug $u = fg$ back into the PDE and separate variables (divide both sides by c^2fg to get the *eigenvalue problem*). Briefly justify why this quantity must be a constant, say λ . Write down an ODE for $g(t)$, and a PDE for $f(x, y)$ (the *Helmholtz equation*). Include four boundary conditions for $f(x, y)$.
- (d) You may assume that $\lambda = -(n^2 + m^2)$, and that the solution to the Helmholtz equation is $f(x, y) = b_{nm} \sin nx \sin my$. Solve the ODE for $g(t)$.
- (e) Find the general solution of the boundary value problem. It will be a superposition (infinite sum) of solutions $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$.

- (f) Find (or better yet, recall from your notes) the Fourier sine series for the following two functions:

$$p(x) = x(\pi - x), \quad \text{defined on } [0, \pi],$$

$$r(x) = 1, \quad \text{defined on } [0, \pi].$$

- (g) Find the particular solution to the initial value problem that additionally satisfies the initial conditions.

- (h) What is the long-term behavior of $u(x, y, t)$, i.e., as $t \rightarrow \infty$. Give a mathematical *and* intuitive (physical) justification.