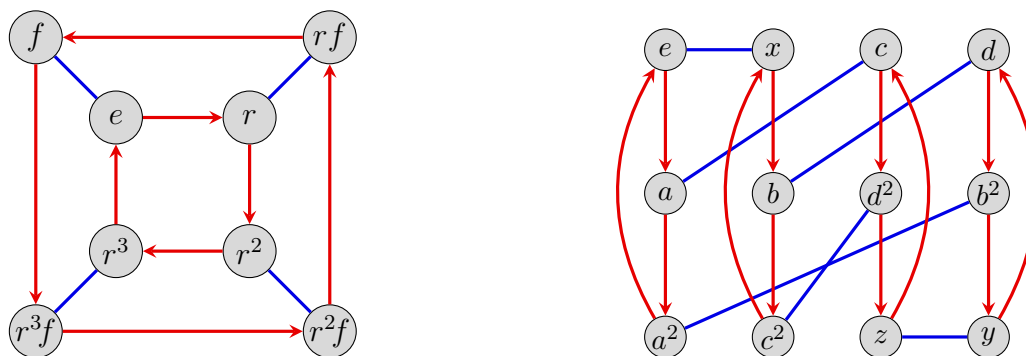


Read the following, which can all be found either in the textbook or on the course website.

- Chapters 3–4 of *Visual Group Theory*, or Chapters 5.1–3 of *IBL Abstract Algebra*
- VGT Exercises 3.5–3.10, 3.12. 4.3–4.5, 4.10–4.14.

Write up solutions to the following exercises.

1. Draw a Cayley diagram for each of the 7 frieze groups using a minimal generating set. Make it clear what your generators are and write out a group presentation for each.
2. Shown below are the Cayley graphs of two groups:  $D_4$  is on the left, and the right is called an “alternating group,” denoted  $A_4$ .



- (a) Create a multiplication table for each group. For consistency, order the elements in  $D_4$  by  $(e, r, r^2, r^3, f, rf, r^2f, r^3f)$  and by  $(e, x, y, z, a, a^2, b, b^2, c, c^2, d, d^2)$  in  $A_4$ .
  - (b) Find the inverse of each element of each group.
  - (c) Write out a group presentation for each group using the generators shown in the Cayley graph.
3. In each of the following multiplication tables, let  $e$  denote the identity element. Complete each table so its depicts a group. There may be more than one way to complete a table, in which case you need to give all possibilities.

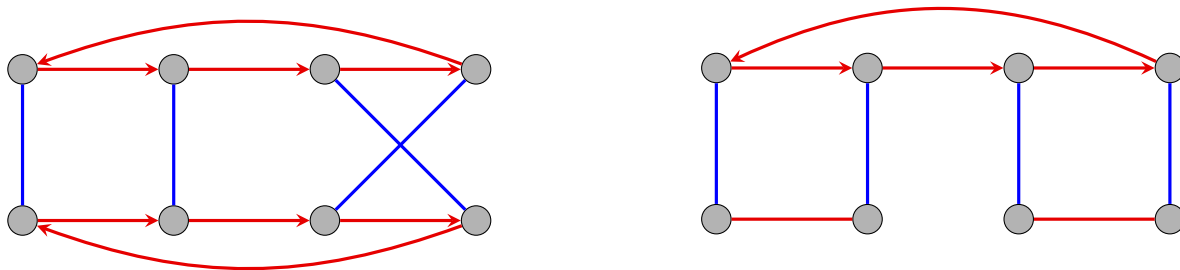
	$e$	$a$
$e$		
$a$		

	$e$	$a$	$b$
$e$			
$a$			
$b$			

	$e$	$a$	$b$	$c$
$e$				
$a$		$e$		
$b$				
$c$				

4. Prove that an element cannot appear twice in the same column of a multiplication table.
5. Using our “unofficial” definition of a group, prove that every group has a unique identity action  $e$ , satisfying  $ge = g = eg$  for every action  $g$  in  $G$ . [Hint: You need to prove both existence and uniqueness. For the latter, assume that  $e$  and  $f$  are both identity actions. Can you prove that  $e = f$ ?]

6. In this exercise, you will see one reason why our unofficial definition of a group doesn't quite match our formal definition. Consider the two diagrams below, neither of which is a valid Cayley diagram.



- Assuming that the arrows represent actions in a group, are the four rules in our unofficial definition of a group satisfied? For each rule, give a brief explanation why or why not.
  - Try to convert each of these diagrams into a multiplication table. What problem arises in each case?
  - Try to write a group presentation describing each “group.” What problem arises in each case?
  - Explain what is wrong with these diagrams that caused the problems you encountered in Parts (b) and (c)?
  - Create another diagram that is not a valid Cayley diagram for the same reason.
7. Let  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$  denote the set of integers, rational numbers, and real numbers, respectively. Let  $\mathbb{Z}^+$ ,  $\mathbb{Q}^+$ , and  $\mathbb{R}^+$  denote the positive integers, rationals, and reals. Let  $\mathbb{Z}^*$ ,  $\mathbb{Q}^*$ , and  $\mathbb{R}^*$  denote the nonzero integers, rationals, and reals.
- Which of the above sets are groups under addition? For each one that is a group (excluding  $\mathbb{R}$ ,  $\mathbb{R}^+$ , and  $\mathbb{R}^*$ ), give a minimal generating set. For each one that is not, give an explicit reason for why it fails.
  - Which of the above sets are groups under multiplication? For each one that is a group (excluding  $\mathbb{R}$ ,  $\mathbb{R}^+$ , and  $\mathbb{R}^*$ ), give a minimal generating set if there is one. For each one that is not, give an explicit reason for why it fails.
  - Let  $n\mathbb{Z}$  denote the set of all integers that are multiples of  $n$ . For what  $n$  is the set  $n\mathbb{Z}$  a group under addition? Give a minimal generating set for each one that is a group.