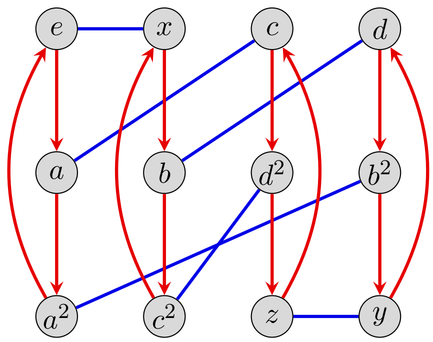
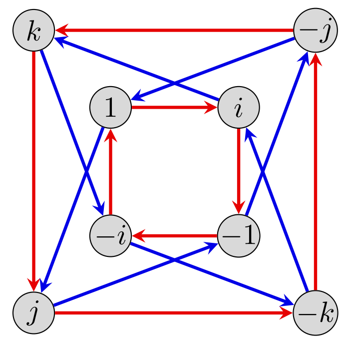
Read the following, which can all be found either in the textbook or on the course website.

- Chapter 5 of *Visual Group Theory*, or Chapter 6 of *IBL Abstract Algebra*.
- VGT Exercises 5.3, 5.5, 5.6–5.9, 5.12, 5.14, 5.16, 5.21, 5.25–5.27, 5.29, 5.36, 5.42, 5.44.

Write up solutions to the following exercises.

1. Carry out the following steps for the groups A_4 and Q_8, whose Cayley graphs are shown below.





   (a) Find the orbit of each element.

   (b) Draw the orbit graph of the group.

2. Prove algebraically that if $g^2 = e$ for every element of a group G, then G must be abelian.

3. Compute the product of the following permutations. Your answer for each should be a single permutation written in cycle notation as a product of disjoint cycles.

   (a) $(1\ 3\ 2)\,(1\ 2\ 5\ 4)(1\ 5\ 3)$ in S_5;

   (b) $(1\ 5)\,(1\ 2\ 4\ 6)\,(1\ 5\ 4\ 2\ 6\ 3)$ in S_6.

4. Write out all $4! = 24$ permutations in S_4 in cycle notation. Additionally, write each as a product of transpositions, and decide if they are even or odd. Which of these permutations are also in A_4?

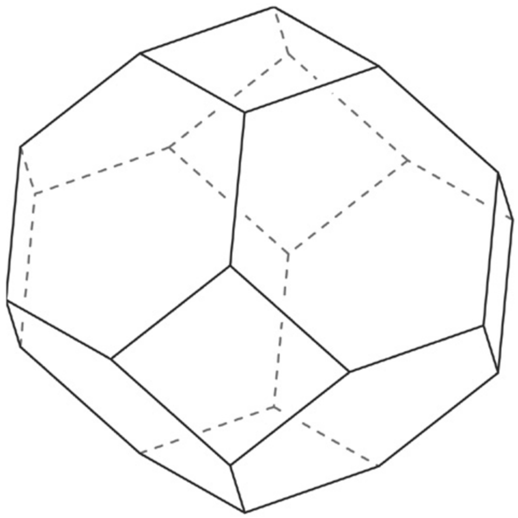
5. (a) The group S_3 can be generated by the transpositions $(1\ 2)$ and $(2\ 3)$. In fact, it has the following presentation

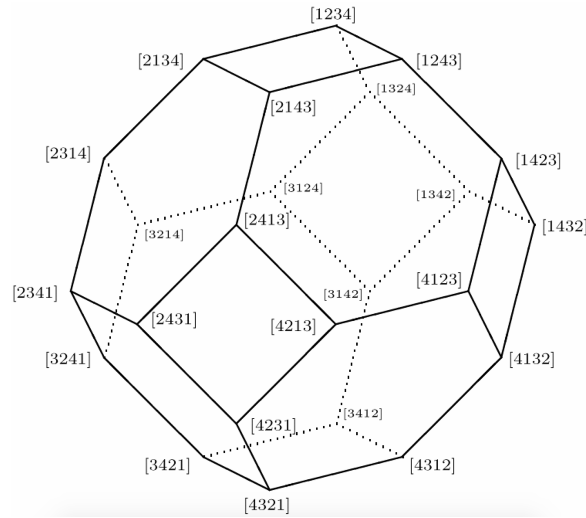
$$S_3 = \langle a, b \mid a^2 = e,\ b^2 = e\,,\ (ab)^3 = e\rangle,$$

   where one can take $a = (1\ 2)$ and $b = (2\ 3)$. Make a Cayley diagram for S_3 using this generating set.

   (b) The group S_4 can be generated by the transpositions $(1\ 2)$, $(2\ 3)$, and $(3\ 4)$. Make a Cayley diagram for S_4 using this generating set. This can be laid out on a polytope called a *permutahedron*, which is a truncated octahedron shown below. On the right, the vertices are labeled with the corresponding permutations in one-line notation.

Make a Cayley graph by labeling the vertices on the unlabeled permutahedron with the 24 permutations of S_4 *in cycle notation*, and color the edges appropriately.





(c) Write down a group presentation for S_4 using the generating set in Part (c).

6. The Cayley diagram for A_4 shown above labels the elements with letters instead of permutations:

$$A_4 = \{e, a, a^2, b, b^2, c, c^2, d, d^2, x, y, z\}.$$

Redraw this Cayley diagram but label the nodes with the 12 even permutations from the previous problem. That is, you need to determine which permutation corresponds to a, which to b, and so on. [*Hint*: There are many possible ways to do this. If you let a be one of the permutations of order 3, and let x be one of the permutations of order 2, then you should be able to determine the remaining elements.]