

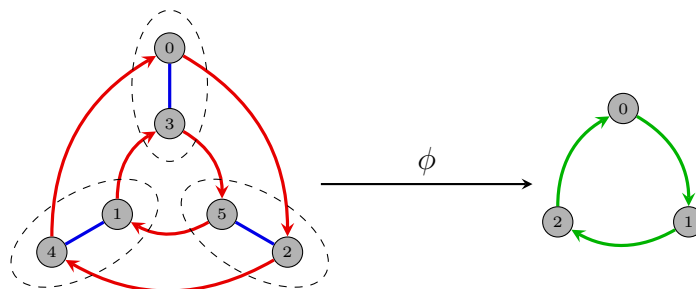
Read the following, which can all be found either in the textbook or on the course website.

- Chapters 8.1–2 of *Visual Group Theory*, or Chapters 4.2, 5.6, 8.1 of *IBL Abstract Algebra*.
- VGT Exercises 8.2, 8.5, 8.6, 8.9, 8.11, 8.13, 8.14.

Write up solutions to the following exercises.

1. Do the following steps for each mapping $\phi: G \rightarrow H$ listed below.
 - (i) Determine whether ϕ is a homomorphism.
 - (ii) Find $\ker \phi := \{g \in G \mid \phi(g) = e\}$ and $\text{im } \phi = \phi(G)$.
 - (iii) Draw Cayley diagrams of the domain and codomain, and arrange them so one can “visually see” the cosets of $\ker \phi$ in G . Draw dotted lines around these cosets.
 - (iv) Is the quotient $G/\ker \phi$ a group? If so, what is it isomorphic to?

Here is an example of Step (iii) for the map $\phi: \mathbb{Z}_6 \rightarrow \mathbb{Z}_3$, defined by $\phi(n) = n \pmod{3}$.



Now, do steps (i)–(iv) for the following maps.

- (a) The map $\phi: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\phi(n) = 4n$.
 - (b) The map $\phi: D_4 \rightarrow \mathbb{Z}_4$ defined by $\phi(r^k f) = k$.
 - (c) The map $\phi: D_4 \rightarrow \mathbb{Z}_4$ defined by $\phi(r^k f) = 2k$.
 - (d) The map $\phi: \mathbb{Z}_4 \rightarrow D_4$ defined by $\phi(k) = r^k$.
 - (e) The map $\phi: D_4 \rightarrow V_4$ defined by $\phi(r) = h$ and $\phi(f) = v$.
2. Prove that $A \times B \cong B \times A$.
 3. For each part below, list *all* homomorphisms with the given domain and codomain.
 - (a) Domain \mathbb{Z}_{15} and codomain \mathbb{Z}_4 .
 - (b) Domain \mathbb{Z}_{412} and codomain \mathbb{Z}_{450} .
 - (c) Domain and codomain both \mathbb{Z}_4 .
 - (d) Domain C_4 and codomain V_4 .
 - (e) Domain and codomain both V_4 .

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4. Prove that there is no embedding $\phi: \mathbb{Z}_n \hookrightarrow \mathbb{Z}$.
5. Let $H \leq G$, and fix $x \in G$. Recall that we showed in class that xHx^{-1} is always a subgroup of G .
- (a) Prove additionally that $xHx^{-1} \cong H$. [*Hint*: Define a mapping from H to xHx^{-1} and prove that it is a homomorphism, one-to-one, and onto.]
 - (b) Use Part (a) to show that $|xy| = |yx|$ for any $x, y \in G$.
6. Let $\phi: G \rightarrow H$ be a homomorphism, and $N \triangleleft H$.
- (i) Show that the set $\phi^{-1}(N) := \{g \in G \mid \phi(g) \in N\}$ is a subgroup of G .
 - (ii) Show that $\phi^{-1}(N)$ is a *normal* subgroup of G .
 - (iii) Show by example that if $M \triangleleft G$, then $\phi(M)$ need not be a normal subgroup of H .