Read the following, which can all be found either in the textbook or on the course website.

- Chapter 9.1 of *Visual Group Theory* (VGT).
- VGT Exercises 8.15–8.18, 9.17.

Write up solutions to the following exercises.

1. The commutator subgroup of a group G is the subgroup

$$G' = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle \,.$$

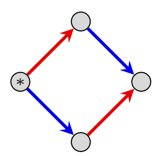
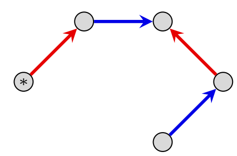
   (a) Prove that G is abelian if and only if $G' = \{e\}$.

   (b) Prove that $G' \lhd G$. [*Hint*: Take a "commutator" $c = aba^{-1}b^{-1}$ and prove that $gcg^{-1} \in G'$.]

   (c) Prove that G' is the intersection of all normal subgroups of G that contain the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$:

$$G' = \bigcap_{C \subseteq N \lhd G} N$$

   (d) If we quotient G by G', then we are in essence, "killing" all non-abelian parts of the Cayley diagram, as shown below:

$ab = ba$  $ab \neq ba$ 

   Prove algebraically that G/G' is indeed abelian.

2. For each of the following groups G, compute its commutator subgroup G' and its abelianization G/G'. Finally, draw the subgroup lattice of G and circle every normal subgroup, and circle twice the one that is G'.

   (a) V_4,

   (b) D_3,

   (c) Q_8.

3. Find the commutator subgroup of each of the following groups and compute its abelianization.

   (a) An abelian group A.

   (b) The alternating group A_n, for $n \geq 5$. [*Hint*: A_n is a *simple group*, which means its only normal subgroups are $\langle e \rangle$ and A_n.]

   (c) The dihedral group D_n for n even.

   (d) The dihedral group D_n for n odd.

4. Recall that the automorphism group of $V_4 = \langle h, v \rangle = \{e, h, v, r\}$, where $r = hv$, has presentation

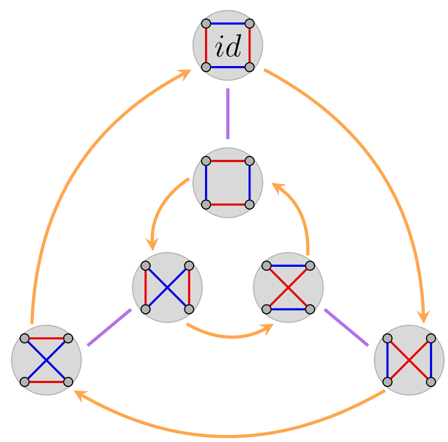
$$\mathrm{Aut}(V_4) = \big\langle \alpha, \beta \mid \alpha^3 = \beta^2 = (\alpha\beta)^2 = id \big\rangle, \quad \text{where} \quad \begin{array}{l} h \overset{\alpha}{\longmapsto} v \\ v \longmapsto r \end{array} \quad \text{and} \quad \begin{array}{l} h \overset{\beta}{\longmapsto} v \\ v \longmapsto h. \end{array}$$

The generating automorphisms are the following permuations of V_4:

$$\alpha : \quad e \quad h \quad v \quad r \qquad \text{and} \qquad \beta : \quad e \quad h \quad v \quad r$$

The multiplication table and Cayley diagram of $\mathrm{Aut}(V_4) = \langle \alpha, \beta \rangle$, which highlights how automorphisms are "re-wirings", are shown below:

	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
id	id	α	α^2	β	$\alpha\beta$	$\alpha^2\beta$
α	α	α^2	id	$\alpha\beta$	$\alpha^2\beta$	β
α^2	α^2	id	α	$\alpha^2\beta$	β	$\alpha\beta$
β	β	$\alpha^2\beta$	$\alpha\beta$	id	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^2\beta$	α	id	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^2	α	id



Repeat the above steps for each of the following groups. Use the Cayley diagram defined by the generating set given. Recall that $\mathrm{Aut}(\mathbb{Z}_n) \cong U_n$.

(a) $\mathbb{Z}_5 = \langle 1 \rangle$,

(b) $\mathbb{Z}_6 = \langle 1 \rangle$,

(c) $\mathbb{Z}_2 \times \mathbb{Z}_3 = \langle (1,0), (0,1) \rangle$, [Recall that $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$]

(d) $\mathbb{Z}_8 = \langle 1 \rangle$.

5. Let G act on a set S. Prove that $\mathrm{Stab}(s)$ is a subgroup of G for every $s \in S$.

6. Suppose the cyclic group C_5 acts on a set $S = \{A, B, C, D\}$.

(a) What are the possible sizes of the orbits?

(b) What are the possible stabilizer subgroups of each element?

(c) Draw the action diagram.

Fully explain your reasoning for each part.