Read the following, which can all be found either in the textbook or on the course website.

- Chapters 10.1–10.5 of *Visual Group Theory* (VGT).
- VGT Exercises 10.1, 10.2, 10.4, 10.5, 10.8–10.14, 10.21, 10.30.

Write up solutions to the following exercises.

1. Recall that a group G is called *simple* if its only normal subgroups are G and $\{e\}$.

   (a) Show that there is no simple group of order $45 = 3^2 \cdot 5$.

   (b) Show that there is no simple group of order pq, where $p < q$ and are both prime.

   (c) Show that there is no simple group of order $12 = 2^2 \cdot 3$.

   (d) Show that there is no simple group of order $56 = 2^3 \cdot 7$.

   (e) Show that there is no simple group of order $108 = 2^2 \cdot 3^3$.

2. The field $\mathbb{Q}(\sqrt[4]{3}, i)$ is called the *splitting field* of the polynomial $f(x) = x^4 - 3$ over $\mathbb{Q}$ because it is the smallest extension field of $\mathbb{Q}$ that contains all roots of $f(x)$.

   (a) Sketch the roots of $f(x) = x^4 - 3$ in the complex plane. Write each one as $a + bi$, where $a, b \in \mathbb{R}$. Additionally, write each root in polar form: $z = Re^{i\theta}$.

   (b) Find a basis for the extension field $\mathbb{Q}(\sqrt[4]{3})$ of $\mathbb{Q}$ and compute its dimension as a $\mathbb{Q}$-vector space. That is, find a minimal set of $v_1, \dots, v_k \in \mathbb{Q}(\sqrt[4]{3})$ such that every $x \in \mathbb{Q}(\sqrt[4]{3})$ can be written as a unique linear combination of the v_i's.

   (c) Is $\mathbb{Q}(\sqrt[4]{3})$ the splitting field of some polynomial $g(x)$ over $\mathbb{Q}$? If yes, find $g(x)$. If no, explain why not.

   (d) Find a basis for $\mathbb{Q}(\sqrt[4]{3}, i) := \mathbb{Q}(\sqrt[4]{3})(i) = \mathbb{Q}(i)(\sqrt[4]{3})$ over each of the fields $\mathbb{Q}(\sqrt[4]{3})$, $\mathbb{Q}(i)$, and $\mathbb{Q}$. What is the dimension of $\mathbb{Q}(\sqrt[4]{3}, i)$ as a vector space over each of these fields?

   (e) $\mathbb{Q}(\sqrt[4]{3}, i)$ is the splitting field of what polynomial over $\mathbb{Q}(\sqrt[4]{3})$? And of what polynomial over $\mathbb{Q}(i)$?

3. Thus far in class, we have seen a number of algebraic extensions of $\mathbb{Q}$, including:

   $$\mathbb{Q}(\sqrt{2}), \quad \mathbb{Q}(\sqrt{3}), \quad \mathbb{Q}(\sqrt{6}), \quad \mathbb{Q}(\sqrt{2}, \sqrt{3}), \quad \mathbb{Q}(i), \quad \mathbb{Q}(\sqrt{-3}, \sqrt[3]{2}), \quad \mathbb{Q}(\sqrt[4]{3}, i), \quad \mathbb{Q}(\sqrt[4]{3}).$$

   Arrange these fields in a subfield lattice, and include $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ as well. Note that there will be (many!) "missing" fields, so only include those listed above. For each edge in this lattice, which corresponds to an extension field $E \supseteq F$, write the degree of the extension of E over F, which by definition is the dimension of E as an F-vector-space.

4. Consider the function

$$\phi\colon \mathbb{Q}(\sqrt{2}) \longrightarrow \mathbb{Q}(\sqrt{2})\,, \qquad \phi(a+b\sqrt{2}) = a - b\sqrt{2}\,.$$

Show that ϕ is a field automorphism, meaning that it satisfies the following equations for all $\alpha, \beta \in \mathbb{Q}(\sqrt{2})$:

$$\phi(\alpha+\beta) = \phi(\alpha) + \phi(\beta)\,, \qquad \phi(\alpha\cdot\beta) = \phi(\alpha)\cdot\phi(\beta)\,.$$

5. Consider the following extension field of $\mathbb{Q}$:

$$K = \mathbb{Q}(\sqrt{2}, i) = \{a + b\sqrt{2} + ci + d\sqrt{2}i \mid a,b,c,d \in \mathbb{Q}\}\,.$$

(a) Find the Galois group $G = \mathrm{Gal}(K)$ of K over $\mathbb{Q}$. For each automorphism $\phi \in G$, write down where it sends the generators $\sqrt{2}$ and i, and then write down

$$\phi(a + b\sqrt{2} + ci + d\sqrt{2}i)\,.$$

(b) Write out a multiplication table for G, and a minimal generating set.

(c) Write down the subfield lattice of K and the subgroup lattice of G. Each subgroup should be expressed by its generators, rather than what subgroup it is isomorphic to.

(d) For each subgroup $H \leq G$, determine the largest subfield of K that H fixes.

(e) For each subfield $F \subseteq K$, determine the largest subgroup of G that fixes F.