1. Carry out the steps outlined below for the following predator–prey model:

$$P_{t+1} = P_t(1 + .8(1 - P_t)) - 4P_tQ_t$$
$$Q_{t+1} = .9Q_t + 2P_tQ_t$$

   (a) Compute the equilbria.

   (b) Use MATLAB and the `twopop` program to make an informed guess as to whether the equilbria are stable or unstable. Print out and attach the phase portrait.

   (c) Linearize the model at each of the equilibria and compute eigenvalues to determine stability.

2. One approach to preventing disease spread is to simply quarantine infectives. Suppose a disease is modeled by the SIR model, but people who get the disease are health-conscience and quarantine themselves. The net result is that a fraction q of the infectives are prevented from having contacts with the susceptibles. Only $1 - q$ of the infectives will be able to spread the disease.

   (a) Modify the equations of the SIR model to reflect this. What value of q gives the usual SIR model?

   (b) Quarantining can be viewed as a way of modifying the transmission coefficient. Suppose an SIR model has transmission coefficient α, and a fraction q of the infectives are successfully quarantined. Then the model with quarantining is identical to a standard SIR model with some other transmission coefficient α', the *effective transmission coefficient*. Give a formula for α' in terms of α and q.

   (c) Use the MATLAB program `sir` to investigate the behavior of your quarantine model for $N = 100$, $\alpha = 0.001$, and $\gamma = 0.05$, and vary q from 0 to 1. Explain the qualitative behavior you see. Can you find a value of q that prevents an epidemic from occurring, regardless of I_0? Estimate the smallest such q.

3. Another approach to preventing disease spread is vaccination of susceptibles. Suppose a public health organization offers a vaccine for a disease modeled by the SIR model. One simple model of this situation counts each successful vaccination in the removed class throughout the duration of the model.

   (a) Suppose that all vaccinations occur before the time $t = 0$. Even if this is not the case, we may assume it is – why?

   (b) Suppose with $N = 100$, we have $I_0 = 1$, with the removed class composed of the fraction q of the population that was successfully vaccinated. Give formulas for S_0 and R_0 (the initial number of recovered people, *not* the basic reproductive number $\mathcal{R}_0$). What value of q gives the usual SIR model?

   (c) Repeat Part (c) from the previous problem for this situation.

4. An isolated island population of 100 individuals is exposed to a particularly deadly disease; an infected individual remains contagious until overcome by death after 4 days. We want to predict the disease's effect on the community on a daily basis. Suppose initially one individual is stricken with the disease.

   (a) What is the removal rate γ?

   (b) For what values of the *relative removal rate* $\rho := \gamma/\alpha$ will an epidemic occur? Use this to determine for what values of the transmission coefficient α an epidemic will occur.

   (c) Use a computer program such as `sir` to estimate the number of days until the epidemic peaks for the values of $\alpha = .003$, $.005$, $.01$, and $.0125$, presenting your data in a table. How does the magnitude of α relate to the time until the peak?

   (d) Calculate the basic reproductive numbers and the relative remove rates ρ for the values of α above, adding that information to your table.

5. The following difference equation is called the *SIS model*:

$$\Delta S = -\alpha SI + \gamma I$$
$$\Delta I = \alpha SI - \gamma I\,.$$

   (a) What disease might be modeled well by the SIS framework?

   (b) Use a computer program such as `sir` or `twopop` to explore the dynamics of the SIS model. Vary the parameters α, γ, N, S_0, and I_0. Describe your findings.

   (c) Solve for all equilibria (S^*, I^*). Are these biologically reasonable? An equilibrium $I^* > 0$ is called an *endemic equilibrium*. Can an SIS disease be endemic?

   (d) Since $S_t + I_t = N$ is constant, substitute $I_t = N - S_t$ back into the formula for S_t and find a formula for S_{t+1} in terms of S_t. Find a formula for I_{t+1} in terms of I_t.

   (e) For the SIR model, the threshold value $\rho := \gamma/\alpha$, called the *relative remove rate*, plays an important role. What does it represent? Is there an analogous threshold value for the SIS model? If so, find it. If not, explain why.

   (f) For the SIR model, the basic reproductive number $\mathcal{R}_0$ plays an important role. How should one define $\mathcal{R}_0$ for the SIS model? Justify your answer.