Read: Chapter 3.1–3.4 of Robeva/Hodge: *Inferring the topology of gene regulatory networks: an algebraic approach to reverse engineering.* By B. Stigler and E. Dimitrova, pages 75–90.

Do:

1. How many Boolean networks $f = (f_1, f_2, f_3)$ fit the following data?

$$(1,1,1) \xrightarrow{f} (1,1,0) \xrightarrow{f} (0,0,1) \xrightarrow{f} (0,0,1)\,.$$

   By inspection, find two of them. Express your answer using Boolean logic and as polynomials in $\mathbb{F}_2[x_1, x_2, x_3]$. Bonus points if one or both of your solutions were found by no one else in the class.

2. Consider the following *time series* in a 3-node polynomial dynamical system over $\mathbb{F}_3$:

$$(1,1,1) \xrightarrow{f} (2,0,1) \xrightarrow{f} (2,0,0) \xrightarrow{f} (0,2,2) \xrightarrow{f} (0,2,2)\,.$$

   For reference, here are the input vectors $\mathbf{s}_i$ and output vectors $\mathbf{t}_i$:

$$\begin{aligned}
\mathbf{s}_1 &= (s_{11}, s_{12}, s_{13}) = (1,1,1)\,, & \mathbf{t}_1 &= (t_{11}, t_{12}, t_{13}) = (2,0,1)\,,\\
\mathbf{s}_2 &= (s_{21}, s_{22}, s_{23}) = (2,0,1)\,, & \mathbf{t}_2 &= (t_{21}, t_{22}, t_{23}) = (2,0,0)\,,\\
\mathbf{s}_3 &= (s_{31}, s_{32}, s_{33}) = (2,0,0)\,, & \mathbf{t}_3 &= (t_{31}, t_{32}, t_{33}) = (0,2,2)\,,\\
\mathbf{s}_4 &= (s_{41}, s_{42}, s_{43}) = (0,2,2)\,, & \mathbf{t}_4 &= (t_{41}, t_{42}, t_{43}) = (0,2,2)\,.
\end{aligned}$$

   (a) Find polynomials f_1, f_2, f_3 in $\mathbb{F}_3[x_1, x_2, x_3]$ that fit the data. That is, $f_j(\mathbf{s}_i) = \mathbf{t}_i$ for all $i = 1,2,3,4$.

   (b) For each $j = 1,2,3,4$, write down the ideal $I_j = I(\mathbf{s}_j)$ of polynomials that vanish on the data point $\mathbf{s}_j$.

   (c) Use the following commands in Macaulay2 to compute the ideal I of polynomials that vanish on *all* of the input data points.

```
R = ZZ/3[x1,x2,x3,MonomialOrder=>Lex];
I = intersect{I1, I2, I3, I4};
```

   Compute a Gröbner basis $\mathcal{G}$ of I.

   (d) Write the *model space* of the time series using your answer to Part (a) as the particular solution.

   (e) Compute the normal form of f_1, f_2, f_3 with respect to $\mathcal{G}$ by reducing them modulo the ideal I. Write the model space using this particular solution.

   (f) Repeat Parts (c)–(e) using `MonomialOrder=>GRevLex`.

   Turn in a print-out of your Macaulay2 worksheet.