$\boxed{\text{Infinity}}$ Mon 8/27

What do we mean by infinity?

Numbers? Lines? Space? Something else?

How does infinity arise in art and architecture?

Can we do math with infinity?

$$\frac{1}{0} = \infty, \quad \frac{-1}{0} = -\infty, \quad \frac{1}{\infty} = 0, \quad \frac{0}{0} = ?$$

$$\infty + \infty = \infty, \quad \infty - \infty = ?? \quad \frac{\infty}{\infty} = ?$$

<u>Bird example:</u> 2 farmers plant 1 seed/day.

A bird eats one seed every 4 days.

				day 4				day 8			
Farmer 1:	1	2	3	~~4~~	5	6	7	~~8~~	9	10	...
Farmer 2	~~1~~	~~2~~	3	4	5	6	7	8	9	10	
	day 4	day 8									

How many seeds are left "at the end of time"?

Are all infinities the same "size"? (What does this even mean?)

e.g.,

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$$

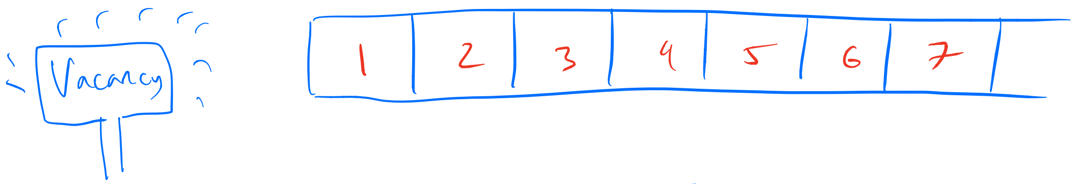
$$2\mathbb{N} = \{2, 4, 6, 8, 10, 12, \ldots\}$$

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

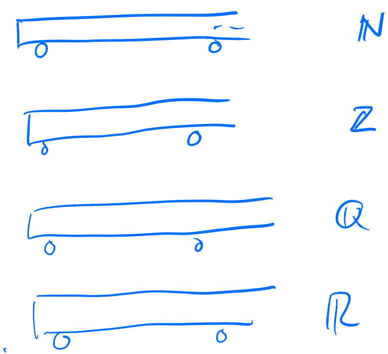
$$\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0,\ \gcd(a,b) = 1\right\}$$

$$\mathbb{R} = \{\text{all real \#s}\}$$

Hilbert's hotel:



Infinite buses arrive...



Proof that $|\mathbb{N}| = |\mathbb{Q}|$

⋮	⋮	⋮	⋮	⋮	
5/1	5/2	5/3	5/4	~~5/5~~	...
4/1	~~4/2~~	4/3	~~4/4~~	4/5	...
3/1	3/2	~~3/3~~	3/4	3/5	...
2/1	~~2/2~~	2/3	~~2/4~~	2/5	...
1/1	1/2	1/3	1/4	1/5	...

Proof that $|\mathbb{Q}| < |\mathbb{R}|$

Suppose we could order them:

Room 1	$\boxed{1}$.0835442863...
Room 2	84.$\boxed{1}$128879923...
Room 3	0.8$\boxed{7}$81446312
Room 4	8.29$\boxed{2}$1313142
Room 5	9.990$\boxed{0}$012382

"Cantor's diagonal argument."

2.2831... doesn't get a room! (why?)

Discuss the continuum hypothesis & Gödel's incompleteness theorem.

Next paradox: We can cover $\mathbb{Q}$ with intervals of total size 1.

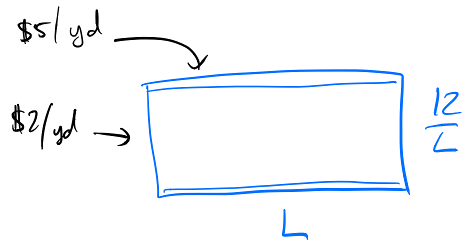
Review of domain, range, and limits

Wed 8/29

Intuitively, $\lim_{x \to c} f(x)$ is "what $f(c)$ should be".

Recall earlier examples.

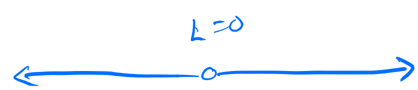
Ex 1: $f(L) = 7L + \frac{48}{L}$



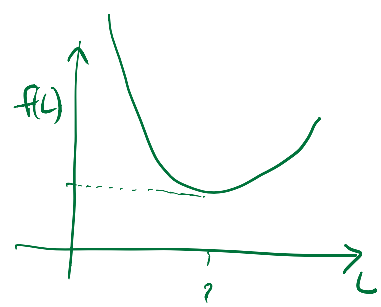
what is the domain?

Ans 1: $L \neq 0$, i.e., $(-\infty, 0) \cup (0, \infty)$

i.e., $L < 0$ or $L > 0$



Ans 2: $L > 0$. [because $L < 0$ is nonsensical]



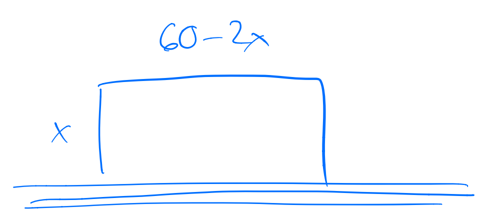
What is the range?

Ans 2: $f(L) > \underset{\sim}{?}$

↖ what we want to find.

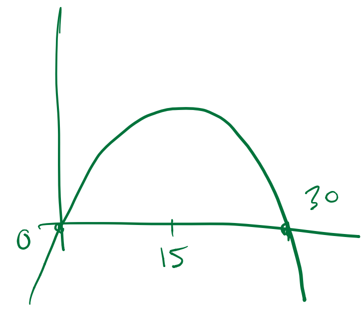
(we'll revisit this)

Ex 2: $g(x) = x(60 - 2x) = 60x - 2x^2$.



Domain: $0 < x < 30$ or $(0, 30)$

or



Range: $(0, 450)$

or $0 < x < 450$.

Ex 1: What "should" $f(0)$ be?

Ans: ∞ (so limit doesn't exist)

We say $\lim_{L \to 0} 7L + \frac{48}{L} = \infty$ (or "doesn't exist")

What "should" $f(1)$ be?

Of course, $f(1) = 7 + 48 = 55$

Ex 2: What "should" $g(0)$ be?

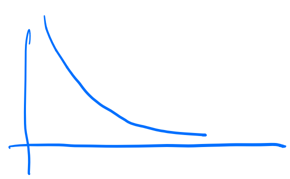
$g(.1) = 5.999\ldots$
$g(.01) = .5999\ldots$
$g(.001) = .0599\ldots$
$\vdots$

$g(0)$ "should be" zero. Write $\lim_{x \to 0} g(0) = 0$.

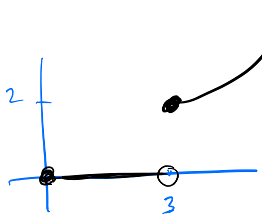
Similarly, $g(30)$ "should be" zero.

Other cases of limits:

$\lim_{x \to \infty} \frac{1}{x} = 0$



We can also define the left-hand limit and right-hand limit.



$f(2)=3$

$$\begin{cases} \lim_{x\to 3^-} f(x) = 0 \\ \lim_{x\to 3^+} f(x) = 2 \end{cases}$$

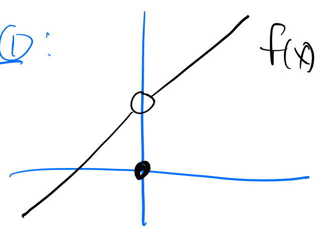
$\lim_{x\to 3} f(x)$ exists iff $\boxed{\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)}$

Caution! Sometimes, the lefthand limit and righthand limits are equal (ie, the limit exists), but...

① They are different from $f(x_0)$

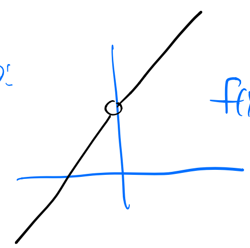
② $f(x_0)$ may not even exist.

Ex ①:



$$f(x) = \begin{cases} x+1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Ex ②:



$f(x) = \dfrac{x^2+x}{x}$.

Limits generally behave like you'd expect:

$$\lim_{x\to c}\left[f(x)+g(x)\right] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$ [provided these exist]

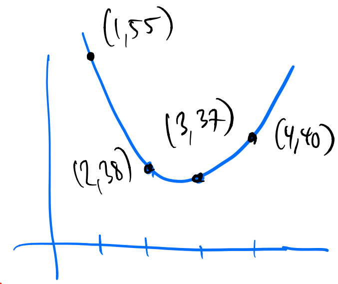
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$ " " "

Ex ②: $\lim_{x\to 0} \dfrac{x^2+x}{x} = \lim_{x\to 0}\left(\dfrac{x^2}{x}\right) + \lim_{x\to 0}\left(\dfrac{x}{x}\right) = \lim_{x\to 0} x + \lim_{x\to 0} 1 = 0+1 = 1$

More on limits

Thurs 8/30

Goal: Determine the lowest/highest point of a curve.



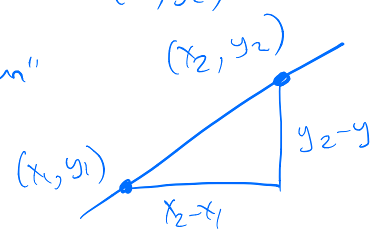
How: At each point P on a curve, we shall seek the line through P that most closely approximates the curve near P. "tangent line"

Key observation: The minimum is where this tangent line is horizontal.

Recall: The slope of the line through (x_1, y_1) and (x_2, y_2) is

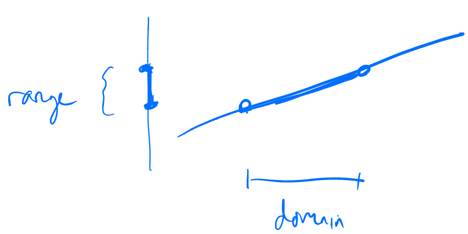
$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{if } x_1 \neq x_2. \qquad \text{"rise over run"}$$

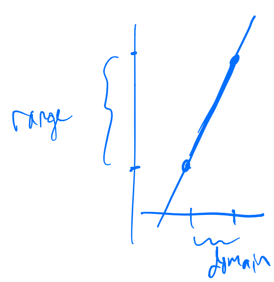
$$\Rightarrow \quad y_2 - y_1 = m(x_2 - x_1)$$



A line is:
- rising if its slope is positive
- falling if its slope is negative
- horizontal if its slope is zero.

Think of slope as a stretching factor





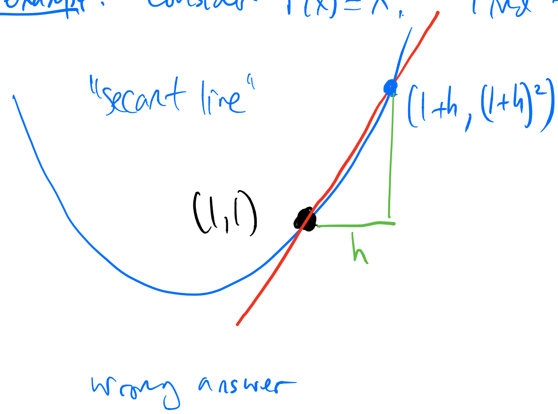
A linear function has the form $f(x) = bx + c$

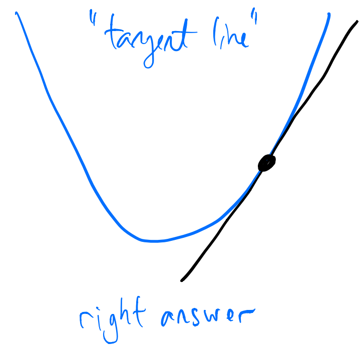
A quadratic function has the form $f(x) = ax^2 + bx + c \quad a \neq 0.$

Next goal: What is the tangent line?

Sherlock Holmes' principle "When you have eliminated the impossible, whatever remains, however improbable, must be the truth."

Simple example: Consider $f(x) = x^2$. Find the slope of the tangent line.





Secant line: Fri. 8/31

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1+h)^2 - (1+h)}{(1+h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h \quad \text{if } h \neq 0.$$

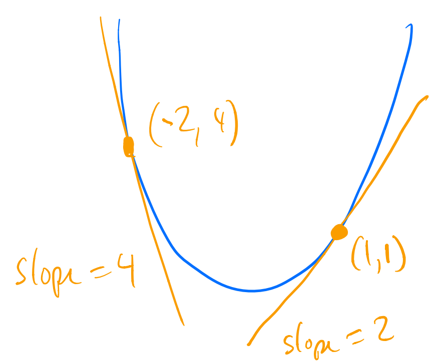
For each $h \neq 0$, this is the wrong answer.

By the "Sherlock Holmes principle", the right answer is when $h = 0$: $\boxed{\text{slope} = 2}$

Let's try again with $y = x^2$ at $P = (-2, 4)$.

$\vdots$

secant line has slope equal to $-4 + h$.

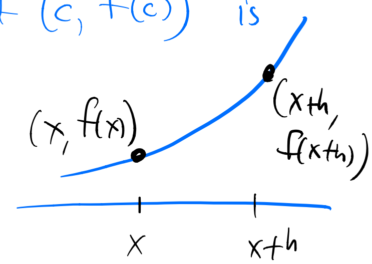


★ How to find a formula for all x?

We need a clear definition of a tangent line at a point.

<u>Def</u>: The slope of a <u>secant line</u> to f at $(c, f(c))$ is

$$\frac{\text{rise}}{\text{run}} = \frac{f(c+h) - f(c)}{(c+h) - c} = \frac{f(c+h) - f(c)}{h}$$



The slope of the tangent line to f at $(c, f(c))$ is:

$$\boxed{\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}}.$$

This determines a new function called the <u>derivative</u> of f.

$f'(x) =$ "slope of the tangent line to f at $(x, f(x))$"

<u>Examples</u>: $f(x) = x^2$ at $(1,1)$ slope $= 2$

$f(x) = x^2$ at $(-2,4)$ slope $= -4$

$f(x) = x^2$ at (x, x^2) slope $= ???$

Use formula: $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x^2} + 2hx + h^2) - \cancel{x^2}}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h) = \boxed{2x}.$$

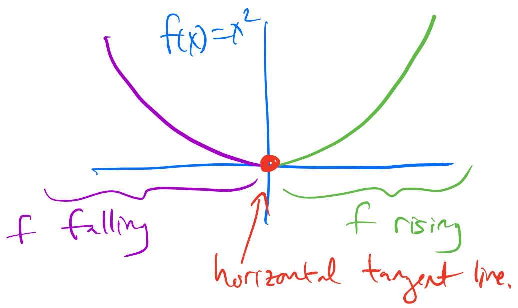
So $\boxed{\text{the derivative of } f(x) = x^2 \text{ is } f'(x) = 2x}$

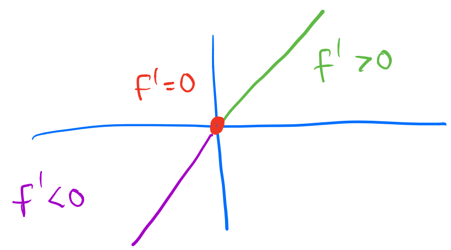
Question: How do we do this for a general function?

Mon. 9/3

The interplay between a function and its derivatives.

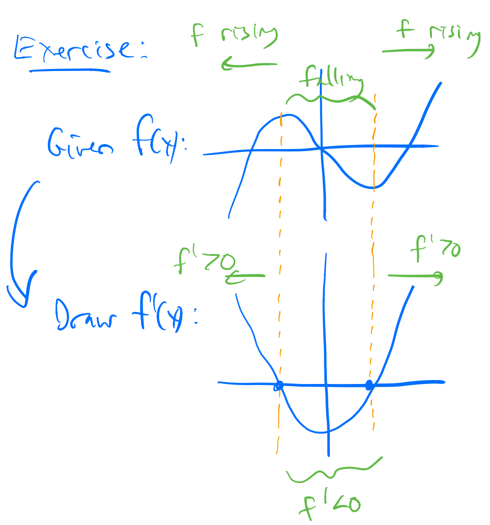
Compare $f(x) = x^2$ vs. $f'(x) = 2x$



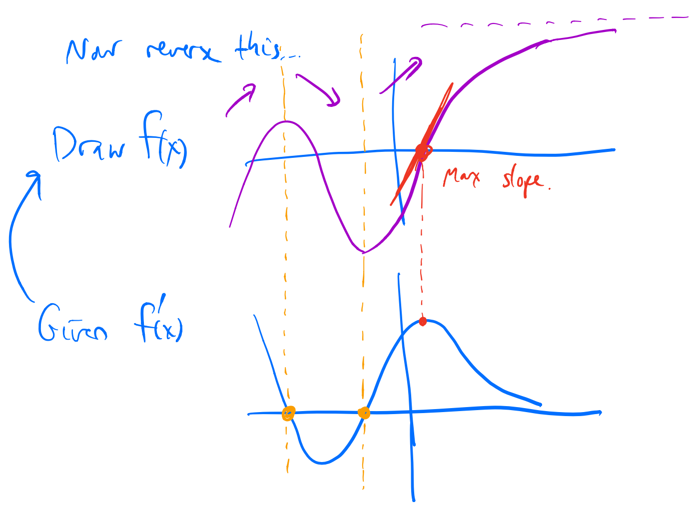


Def: If the tangent line at $(c, f(c))$ is horizontal, then c is a critical point.

Exercise:



Now reverse this...



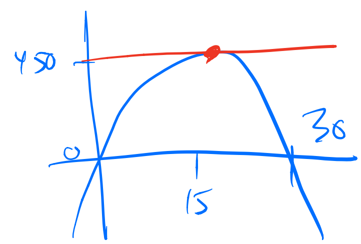
Application: let's maximize $g(x) = x(60-2x) = 60x-2x^2$

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{[60(x+h)-2(x+h)^2]-[60x-2x^2]}{h}$$

$$= \lim_{h\to 0} \frac{[\cancel{60x}+60h-2(\cancel{x^2}+2xh+h^2)]-[\cancel{60x}-\cancel{2x^2}]}{h}$$

$$= \lim_{h\to 0} \frac{60h-4xh-2h^2}{h} = \lim_{h\to 0} 60-4x-2h = \boxed{60-4x}$$

So $g'(x) = 60-4x$



set equal to zero:

$g'(x) = 60-4x = 0 \Rightarrow x = 15$

So g has a horizontal tangent line at $x=15$.

This is where g is maximized: $g(15) = 15(60-30) = 450$.

Wed 9/5

Notation: Write $(f(x))'$ for $f'(x)$.

Last time, we saw that $(60x-2x^2)' = 60-4x$

Goal: Find formulas for the derivative of functions, e.g.,

$(x^n)' =$ $\qquad$ $(f+g)' =$

$(\sin x)' =$ $\qquad$ $(cf)' =$

$(e^x)' =$ $\qquad$ $(fg)' =$

$(\ln x)' =$ $\qquad$ $\left(\frac{f}{g}\right)' =$

- **Ex:** let $f(x) = x^n$, for some non-negative integer n.

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{[\cancel{x^n} + n x^{n-1} h + \_\_ x^{n+2} h^2 + \dots + h^n] - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} n x^{n-1} + h(\ \dots\dots\ ) = n x^{n-1}.$$

- **Derivative of sums:**

$$(f+g)' = \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

- **Derivatives of scalar multiplication:**

$$(cf)' = \lim_{h\to 0} \frac{c f(x+h) - c f(x)}{h}$$

$$= \lim_{h\to 0} c \left[\frac{f(x+h) - f(x)}{h} \right] = c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = c f'(x).$$

Application: derivatives of polynomials.

$$(x^5 + 4x^3 - 2)' = 5x^4 + 12x^2.$$

Fri 9/7

- Reciprocal rule:

$$\left(\frac{1}{f}\right)' = \lim_{h\to 0}\left[\frac{1}{f(x+h)} - \frac{1}{f(x)}\right]h$$

$$= \lim_{h\to 0}\left[\frac{f(x)}{f(x+h)\,f(x)} - \frac{f(x+h)}{f(x+h)\,f(x)}\right]\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{f(x)-f(x+h)}{f(x+h)\,f(x)}\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-[f(x+h)-f(x)]}{h}\cdot\frac{1}{f(x+h)\,f(x)}$$

$$= -f'(x)\cdot\frac{1}{f(x)\,f(x)} = \frac{-f'(x)}{(f(x))^2}$$

Example: Compute $(x^{-3})' = \left(\frac{1}{x^3}\right)'$.

Let $f(x) = x^3$ $\quad \left(\frac{1}{f}\right)' = \frac{-f'(x)}{(f(x))^2} = \frac{-3x^2}{(x^3)^2} = \frac{-3x^2}{x^6} = \boxed{-3x^{-4} = \frac{-3}{x^4}}$

This generalizes further.

Example: Compute $(x^{-n})' = \left(\frac{1}{x^n}\right)'$.

Let $f(x) = x^n$ $\quad \left(\frac{1}{f}\right)' = \frac{-f'(x)}{(f(x))^2} = \frac{-nx^{n-1}}{(x^n)^2} = \frac{-nx^{n-1}}{x^{2n}} = \boxed{-nx^{-n-1} = \frac{-n}{x^{n+1}}}$

$f'(x) = nx^{n-1}$

Example: $\left(\frac{1}{x^2+1}\right)' = \frac{-(x^2+1)'}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2}$

Remark: $(x^n)' = nx^{n-1}$ holds for all integers n. (positive & negative)

- Product rule.

$$(fg)' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

- Quotient rule. Compute $\left(\frac{f}{g}\right)'$. [It's not f'/g'!]

Key point: $\frac{f}{g} = f \cdot \frac{1}{g}$. We'll use the product and reciprocal rules.

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)'$$

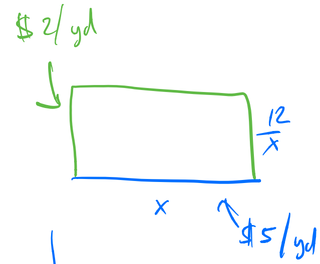
$$= \frac{f'}{g} + f \cdot \frac{-g'}{g^2}$$

$$= \frac{f'g}{g^2} - \frac{fg'}{g^2} = \boxed{\frac{f'g - fg'}{g^2}}$$

Example: $$\left(\frac{x^2+1}{x^3-2x^2+3}\right)' = \frac{2x(x^3-2x^2+3) - (x^2+1)(3x^2-4x)}{x^3-2x^2+3}$$

Mon 9/10

Return to old example: $f(x) = 7x + \frac{48}{x} = 7x + 48x^{-1}$

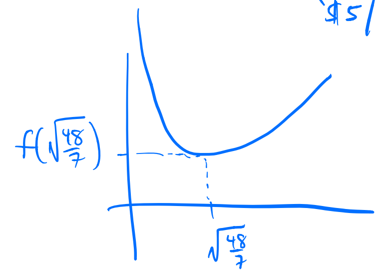


Goal: Find min. of $f(x)$ (this is min cost).

$$f'(x) = 7 - 48x^{-2} = 0$$

$$7 - \frac{48}{x^2} = 0 \implies x^2 = \frac{48}{7}$$

$$\implies x = \pm\sqrt{\frac{48}{7}} \approx 2.619\ldots$$



The min cost of the fence is $f\left(\sqrt{\frac{48}{7}}\right) = 7\sqrt{\frac{48}{7}} + 48/\sqrt{\frac{48}{7}} \approx 36.661$

Next goal: Compute derivatives of trig functions ($\sin x$, $\cos x$, etc.)

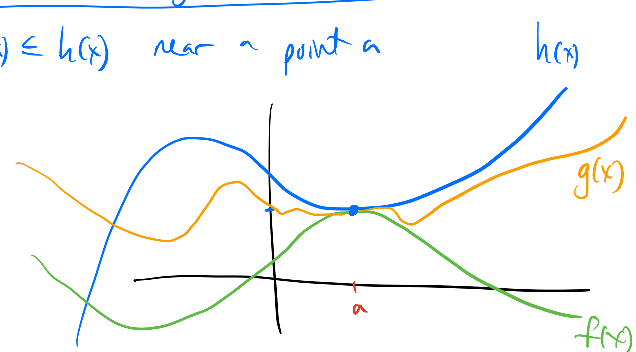
To do this, we'll encounter $\lim_{h \to 0} \frac{\sin h}{h}$.

To evaluate these, we'll need the following:

Squeeze theorem: Suppose $f(x) \le g(x) \le h(x)$ near a point a

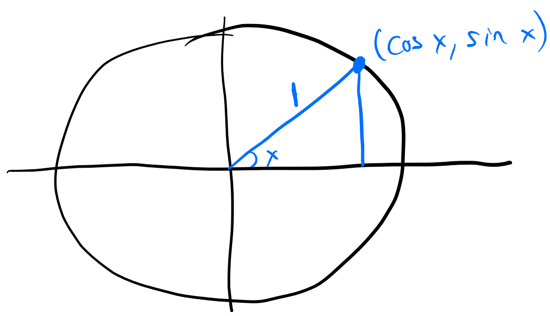
If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$,

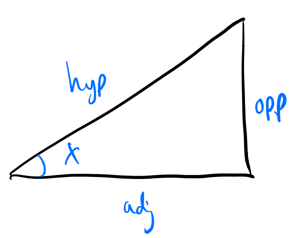
then $\lim_{x \to a} g(x) = L$.



Wed 9/12

Quick review of trig functions



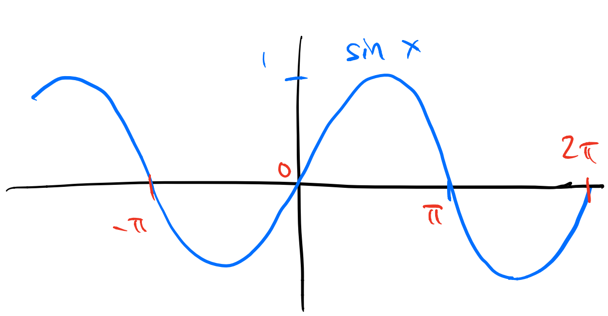


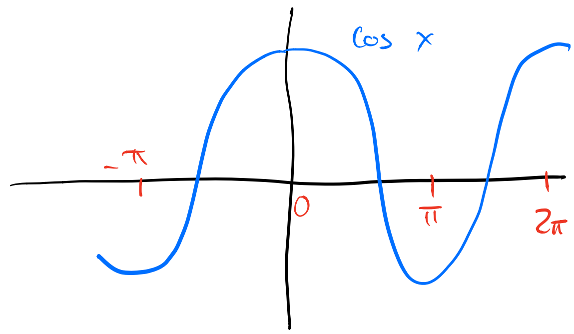
$$\sin x = \frac{\text{opp}}{\text{hyp}}$$

$$\cos x = \frac{\text{adj}}{\text{hyp}}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\text{opp}}{\text{adj}}$$

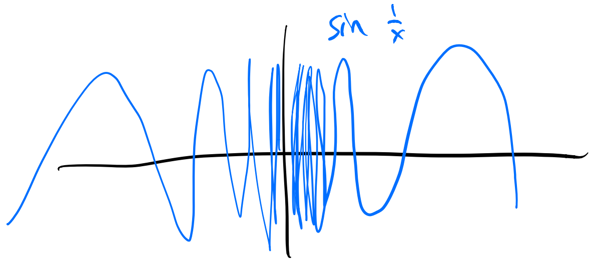
$\frac{1}{\sin x} = \csc x$, $\frac{1}{\cos x} = \sec x$, $\frac{1}{\tan x} = \cot x$



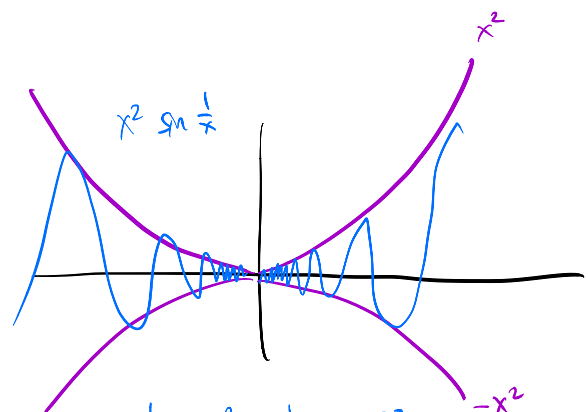


Application of squeeze theorem:

Compute $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$



$\lim_{x\to 0} \sin \frac{1}{x}$ DNE



$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$??

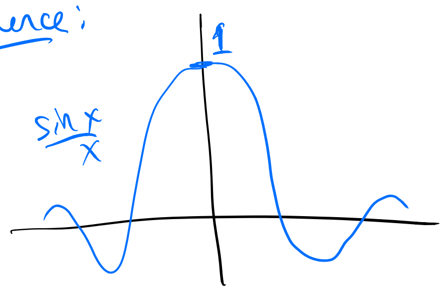
Note: $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$

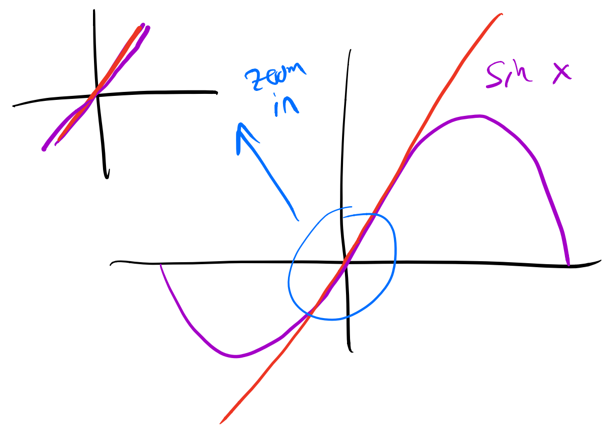
as $x \to 0$ ↓ 0 | as $x \to 0$ by squeeze thm ↓ 0 | as $x \to 0$ ↓ 0

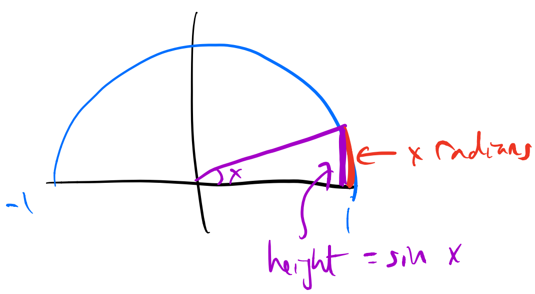
Next: Compute $\lim_{x\to 0} \frac{\sin x}{x}$.

What do you think this should be? [I claim it's 1]

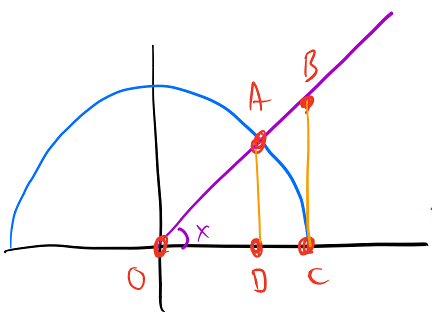
Evidence:







The squeeze theorem confirms it.



area ΔOAD < area sector OAC < area ΔOBC

$$\Rightarrow \frac{1}{2}\cos x \sin x < \frac{x}{2} < \frac{1}{2}\tan x$$

(mult. by $\frac{2}{\sin x}$) $\Rightarrow \cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$

take reciprocal $\Rightarrow \frac{1}{\cos x} > \frac{\sin x}{x} > \cos x$

Apply squeeze thm. (as $x \to 0$) $\Rightarrow$ $\frac{1}{\cos x} \to 1$, $\frac{\sin x}{x} \to 1$ by sq. thm., $\cos x \to 1$

Fri 9/14

Exercise: Use the squeeze theorem to verify that

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$$

Let $f(x) = \sin x$.

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\sin h \cos x}{h}$$

$$= (\sin x) \lim_{h\to 0} \frac{\cos h - 1}{h} + (\cos x) \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \boxed{\cos x}$$

A similar exercise can verify that $(\cos x)' = -\sin x$.

We can use the quotient rule to compute the derivative of the other trig functions.

Ex: $$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

Summary

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$

Note: The 2^{nd} column can be gotten from the 1^{st} column by adding/removing "co" and a negative sign.

Next big idea: The derivative can be viewed as a way to measure the instaneous rate of change of a function.

Notations (1600's): Lagrange: $f'(x)$

Euler: Df

Newton: $\dot{y}$ adopted in Britian

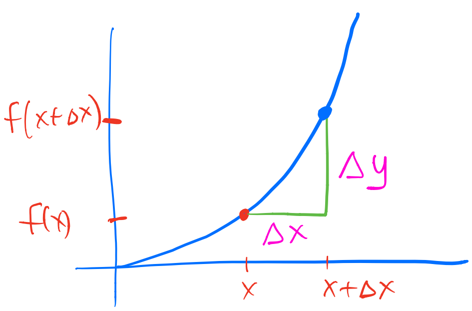
☆ Leibniz: $\frac{dy}{dx}$ adopted in Europe.

Advantages of Leibniz's notation resulted in Britian falling 100-200 hundred years behind mainland Europe mathematically.

Mon 9/17

Recall Leibniz's notation for the derivative: if $y = f(x)$, then $\frac{dy}{dx} = f'(x)$.

Motivating example: Let $y = x^2$



Slope of secant line is $\frac{\Delta y}{\Delta x}$.

$$\Delta y = (x+\Delta x)^2 - x^2$$
$$= x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2$$
$$= 2x(\Delta x) + (\Delta x)^2$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = 2x(\Delta x) + \Delta x, \quad \boxed{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x}$$

Leibniz defined $\boxed{\dfrac{dy}{dx} := \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}}$

Notation reasoning: Ancient Greek symbol Δ had evolved "in the limit" to the modern "d".

Practice this notation:

If $y = x^2$, then $\dfrac{dy}{dx} = \dfrac{d}{dy}(x^2) = 2x$

$A = s^2$, then $\dfrac{dA}{ds} = \dfrac{d}{ds}(s^2) = 2s$

$f(x) = x^3$, then $\dfrac{df}{dx} = 3x^2$

Product rule: If $y = f \cdot g$, then $\dfrac{dy}{dx} = \dfrac{df}{dx} \cdot g + f \cdot \dfrac{dg}{dx}$

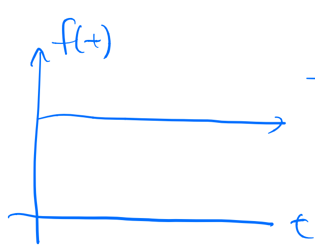
Reciprocal rule: $\dfrac{d}{dx}\left(\dfrac{1}{f}\right) = -\dfrac{1}{f^2} \cdot \dfrac{df}{dx}$

Quotient rule: $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{1}{g^2}\left(\dfrac{df}{dx} \cdot g - f \cdot \dfrac{dg}{dx}\right)$.

Let $f(t)$ = position of an object at time t.

Examples:

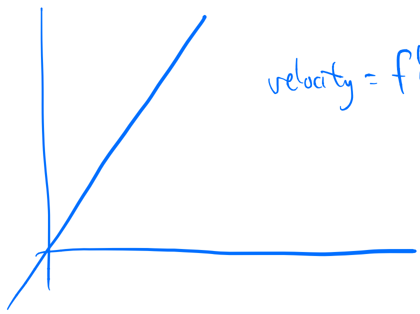
- $f(t) = 30$ mi.



$f'(t) = 0 \ \frac{mi}{hr}$.

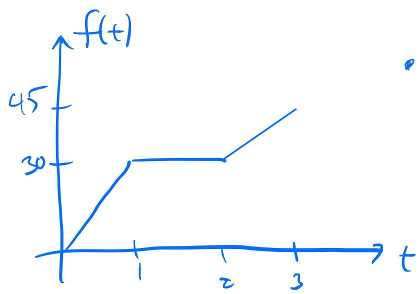
★ velocity is rate of change of position

- $f(t) = 60t$.



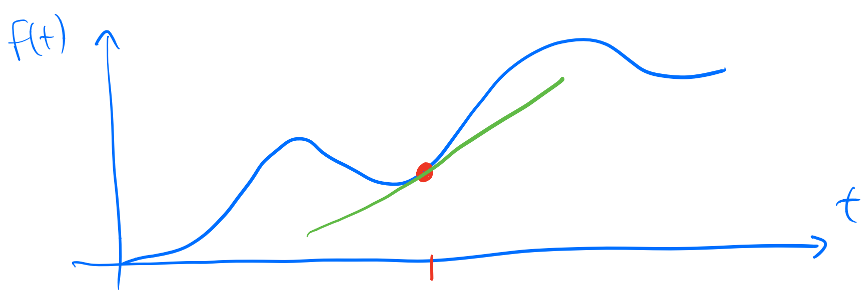
velocity $= f'(t) = 60 \frac{mi}{hr}$

High school math:



- velocity of a piecewise function is just the slope of that particular piece.

Calculus: Suppose the function $f(t)$ is no longer piecewise



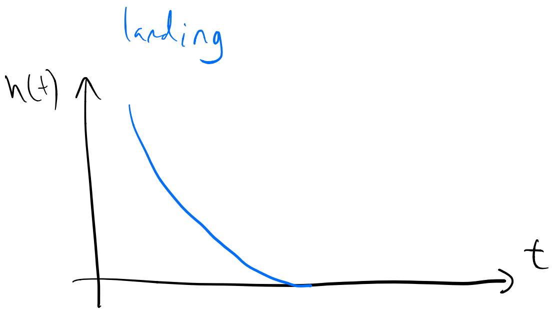
How fast are you traveling at this point in time?

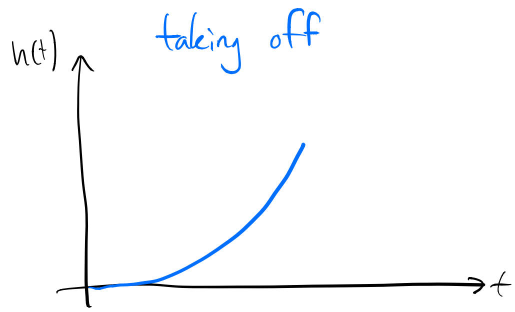
"instantaneous velocity"

Things to ponder about infinitesimals and calculus...

- Can you ever go from motion to non-motion instantaneously (i.e., without "hitting a brick wall"), or vice-versa.
- Actually, that has to happen, any time you come to a complete stop

Consider the height of an airplane:





is there:

- a last point of time in the air?
- a first point of time on the ground?
- a last point of time on the ground?
- a first point of time in the air?

Wed 9/19

Let $h(t) = -16t^2 + 64$ be the height of a ball thrown in the air.

Questions:

ⓐ What is the average speed during the 1^{st} second of flight?

ⓑ What is the average speed during the 2^{nd} second of flight?

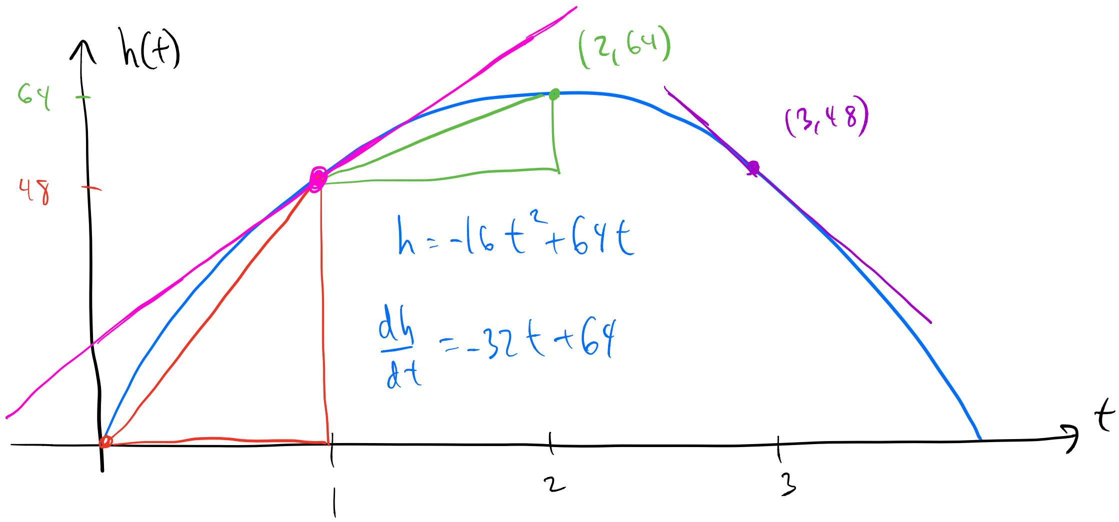
ⓒ What is the instantaneous speed at $t=1$?

ⓓ When $t=3$, is the ball going up or down?

ⓔ When does the ball reach its max height?

ⓕ What is the rock's initial velocity?

ⓖ What is the rock's acceleration.



(a) $\frac{h(1)-h(0)}{1-0} = \frac{48-0}{1-0} = 48$ ft

(b) $\frac{h(2)-h(1)}{2-1} = \frac{64-48}{2-1} = 16$ ft

(c) $\lim_{\Delta x \to 0} \frac{\Delta h}{\Delta t} = \left.\frac{dh}{dt}\right|_{t=1} = h'(1) = 32 \frac{ft}{sec}$.

(d) Down, because $h'(3) < 0$.

(e) $\frac{dh}{dt} = -32t + 64 = 0 \Rightarrow t = 2$ sec.

(f) $h'(0) = 64 \frac{ft}{sec}$

(g) Acceleration is <u>rate of change of velocity</u>

<u>position:</u>	<u>velocity</u>	<u>acceleration</u>
$h(t)$	$v(t) = \frac{dh}{dt} = h'(t)$	$a(t) = v'(t) = \frac{dv}{dt} = h''(t)$.

Note that $h''(t) = -16 \frac{ft}{sec}/sec = -16 \frac{ft}{sec^2}$.

1st derivative: $h'(t) > 0 \Rightarrow$ ball traveling up

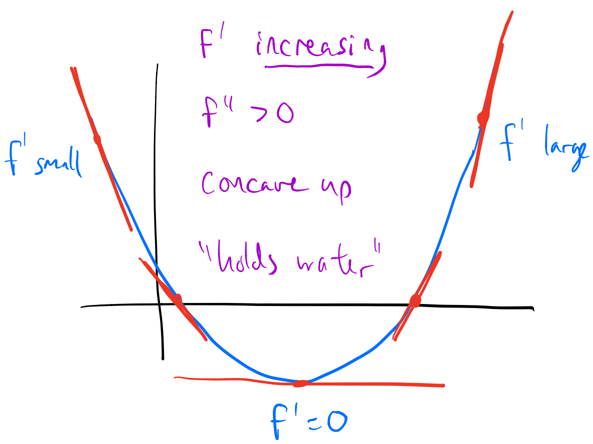
$h'(t) < 0 \Rightarrow$ ball traveling down

2nd derivative: $h''(t) > 0 \Rightarrow$ ball speeding up (vel. is increasing)

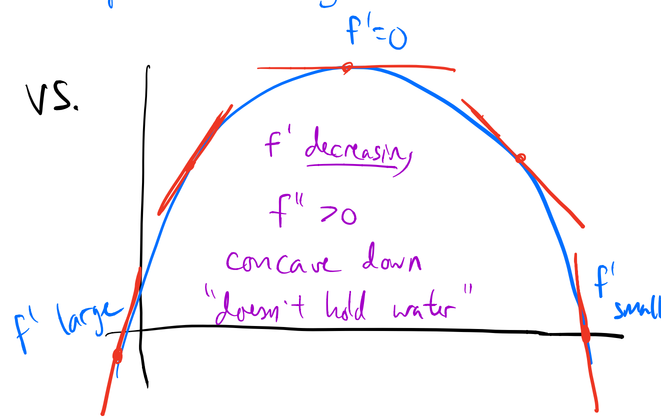
$h''(t) < 0 \Rightarrow$ ball slowing down (vel. is decreasing).

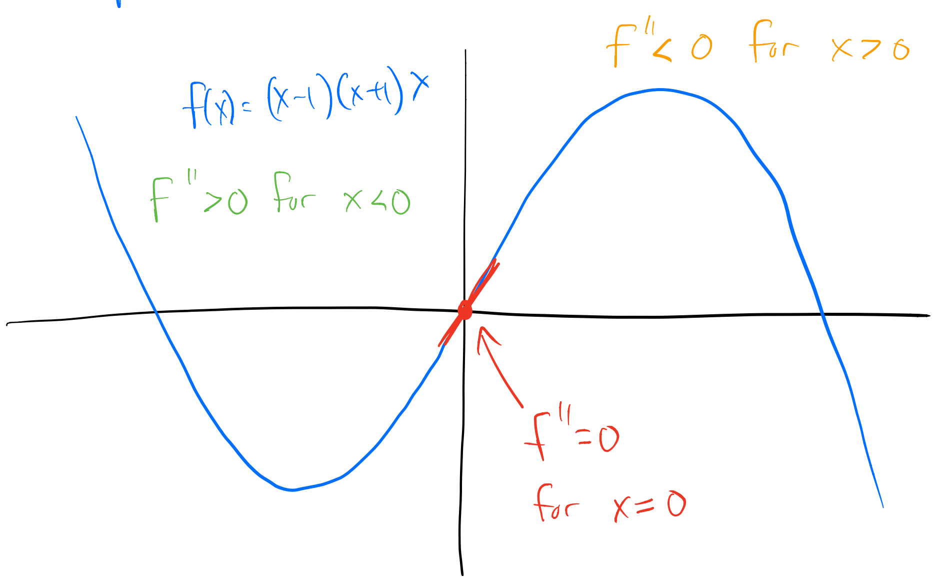
What does the second derivative tell us:

$f''(x) = (f'(x))'$ = rate of change of the slope of the tangent line



vs.

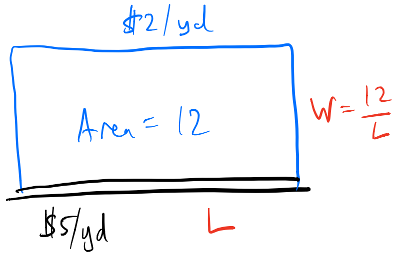




Fri 9/21

Chain rule

Old example:



cost $C = 7L + \frac{48}{L}$ "C is a function of L"

length $L = \frac{12}{w}$ "L is a function of W"

$$C = 7\left(\frac{12}{w}\right) + \frac{48}{12/w}$$

$$= \frac{84}{w} + 4W$$ "C is a function of W"

Question: How are the derivatives $\frac{dC}{dL}$, $\frac{dL}{dW}$, $\frac{dC}{dW}$ related?

Analogy: Suppose Clemson scores 3x as much as GT

Suppose GT scores 2x as much as USC.

Question: How much more does Clemson score as USC?

Answer: 6x.

$$\frac{d\,\text{Clemson}}{d\,\text{GT}} \cdot \frac{d\,\text{GT}}{d\,\text{USC}} = \frac{d\,\text{Clemson}}{d\,\text{USC}}$$

$$3 \cdot 2 = 6.$$

Chain rule: Given $f(g(x))$,

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} \qquad \left[f(g(x))\right]' = f'(g(x)) \cdot g'(x)$$

new notation old notation

Practice:

Ex 1: Let $y = (3x^2+7x)^5$

Then $y = u^5$, where $u = 3x^2+7x$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 5u^4\cdot(6x+7) = 5(3x^2+7x)^4(6x+7).$$

OR: $f(x) = \left(\boxed{3x^2+7x}\right)^5$

$$f'(x) = 5\left(\boxed{}\right)^4\cdot\frac{d}{dx}\left(\boxed{}\right)$$

$$= 5(3x^2+7x)^4\cdot(6x+7)$$

Ex 2: $y = \sin(3x^2+5x-7)$

$$\frac{dy}{dx} = \cos(3x^2+5x-7)\cdot\frac{d}{dx}(3x^2+5x-7)$$

$$= \cos(3x^2+5x-7)\cdot(6x+5)$$

Ex 3: $y = \sin(2x)$

$$\frac{dy}{dx} = \cos(2x)\cdot\frac{d}{dx}(2x) = 2\cos(2x)$$

More generally:
$$\frac{d}{dx}(\sin kx) = k\cos kx$$

$$\frac{d}{dx}(\cos kx) = -k\sin kx.$$

Mon 9/24

Implicit differentiation

Sometimes, a function $y = y(x)$ is defined <u>implicitly</u>, rather than explicitly.
Even in these cases, we can still find the derivative, $y' = \frac{dy}{dx}$.

<u>Ex:</u> Consider a function y defined by $xy + x\sin y = 3x$

<u>Method:</u> Differentiate both sides.

$$xy + x\sin y = 3x$$

$$(xy)' + (x\sin y)' = 3x$$

$$1\cdot y + xy' + 1\cdot \sin y + x(\cos y)\cdot y' = 3$$

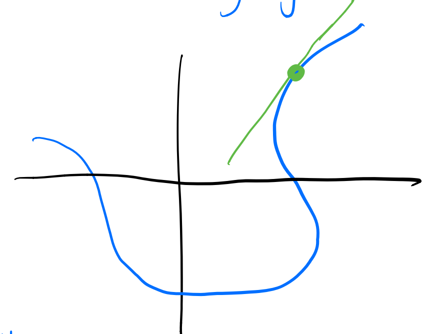
<u>Recall:</u> $y' = \frac{dy}{dx}$

$$y'(x + x\cos y) + y + \sin y = 3$$

Collect y'

$$\boxed{y' = \frac{3 - y - \sin y}{x + x\cos y}}$$

<u>Ex:</u> Find the equation of the line tangent to $x^2 + xy - y^3 = 7$ at the point $(x_0, y_0) = (3, 2)$.



$$\frac{d}{dx}(x^2 + xy - y^3) = \frac{d}{dx}7$$

$$2x + 1\cdot y + x\cdot\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x - 3y^2) = -2x - y \Rightarrow \frac{dy}{dx} = \frac{-2x - y}{x - 3y^2}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,2)} = \frac{-2(3) - 2}{3 - 3\cdot 2^2} = \frac{-8}{-9} = \frac{8}{9}.$$

<u>Tangent line:</u> $\boxed{y - 2 = \frac{8}{9}(x - 3)}$

Wed. 9/26

Application of implicit differentiation: derivative of $x^{p/q}$.

☆ How to compute $\frac{d}{dx}\left(x^{p/q}\right)$?

Write $y = x^{p/q}$

$\Rightarrow y^q = x^p$ $\qquad\qquad y^{q-1} = \frac{y^q}{y} = \frac{x^p}{x^{p/q}} = x^{p-p/q}$

$\frac{d}{dx}(y^q) = \frac{d}{dx}(x^p)$

$q y^{q-1} \frac{dy}{dx} = p x^{p-1}$

$\Rightarrow \frac{dy}{dx} = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q} x^{(p-1)-(p-\frac{p}{q})}$

$= \frac{p}{q} x^{\frac{p}{q}-1}$

Thus, the power rule $\boxed{\frac{d}{dx} x^n = n x^{n-1}}$ also works for any rational number n.

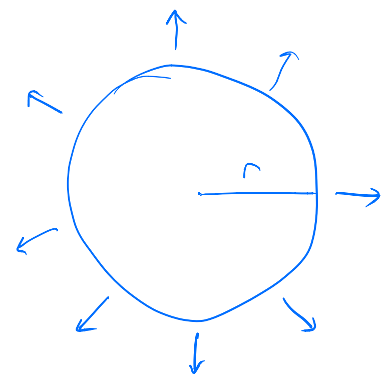
Examples:

- $\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}\sqrt[3]{x^2+3x-1} = \frac{d}{dx}\left(x^2+3x-1\right)^{1/3} = \frac{1}{3}\left(x^2+3x-1\right)^{-2/3} \cdot (2x+3)$

Related rates.

$\boxed{\text{Example}}$ A rock is dropped in a pond, and the ripple expands at a rate of 3 in/sec. How fast is the area increasing when the radius is 7 in?

Given info: $\frac{dr}{dt} = 3 \ \frac{in}{sec}$



$A = \pi r^2$

Want: $\left.\frac{dA}{dt}\right|_{r=7}$

Note: We have functions $A(r)$ and $r(t)$.

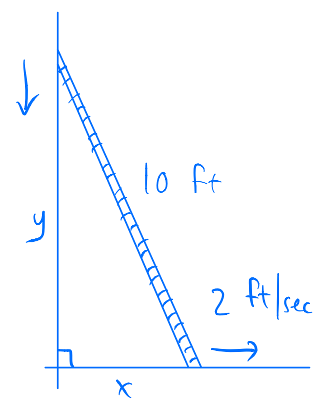
Easy to mess up: $\frac{d}{dt} A(r(t)) = A'(r) \cdot r(t) = 2\pi r \cdot 3 = 6\pi r.$

Better: $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r \cdot 3 = 6\pi r$

So $\left.\frac{dA}{dt}\right|_{r=6} = 6\pi r.$

Note: This is different from the optimization problems we saw earlier, since we're not trying to minimize or maximize anything.

Example: A 10-ft ladder rests against a wall. If the base is pulled away at a rate of 2 ft/sec, how fast is the top of the ladder falling when the ladder is 6 ft from the wall?

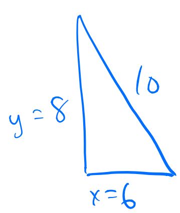


Given info: $\frac{dx}{dt} = 2 \ \frac{ft}{sec}$, $\quad x^2 + y^2 = 100$

Find: $\left.\frac{dy}{dt}\right|$ $\qquad x(t)$ and $y(t)$.

$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(100)$

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$



$x^2 + y^2 = 100$

$6^2 + y^2 = 100$

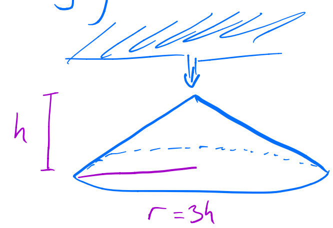
$y^2 = 64 \Rightarrow y = 8.$

$2 \cdot 6 \cdot 2 + 2 \cdot 8 \cdot \frac{dy}{dt} = 0 \Rightarrow 16 \frac{dy}{dt} = 24 \Rightarrow \boxed{\frac{dy}{dt} = 1.5 \ ft/sec}$

Fri 9/28

Another related rates problem.

Sandpile: Sand falls from an overhead bin. It forms a sandpile with a radius that is 3 times its height. If it falls at a rate of 120 ft^3/min, how fast is the height changing when the pile is 10 ft high?



Given info: $r = 3h$

$$\frac{dV}{dt} = 120 \ \text{ft}^3/\text{min}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (3h)^2 \cdot h = 3\pi h^3$$

Find: $\left.\frac{dh}{dt}\right|_{h=10}$.

Chain rule: $\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$$V = 3\pi h^3 \Rightarrow \frac{d}{dV}(V) = \frac{d}{dV}(3\pi h^3)$$

$$\Rightarrow 1 = 27\pi h^2 \cdot \frac{dh}{dV} \Rightarrow \frac{dh}{dV} = \frac{1}{27\pi h^2} = \frac{1}{2700\pi} \text{ when } h=10$$

So $\left.\frac{dh}{dt}\right|_{h=10} = \left.\frac{dh}{dV}\right|_{h=10} \cdot \left.\frac{dV}{dt}\right|_{h=10} = \frac{1}{2700\pi} \cdot 120 = \boxed{\frac{12}{270\pi} \frac{\text{ft}}{\text{min}}}$

Mon 10/3

Exponential functions

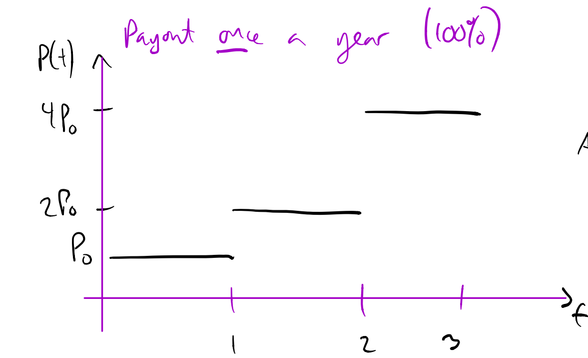
We hear a lot "e is a number that comes up a lot in nature."

But what does that mean?

Motivating example:

Consider an investment that grows at a 100% rate.

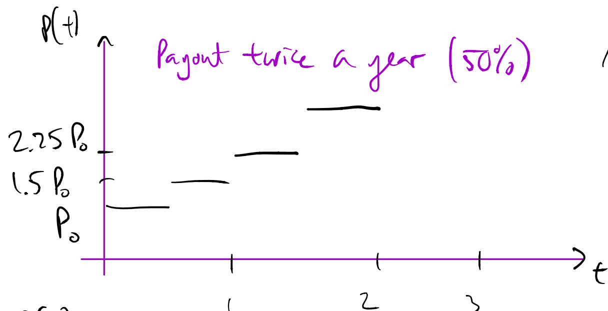
For simplicity, suppose the interest rate is 100%.



rate = 100%

After 1 year: $P(1) = P_0(1 + 1) = 2P_0$

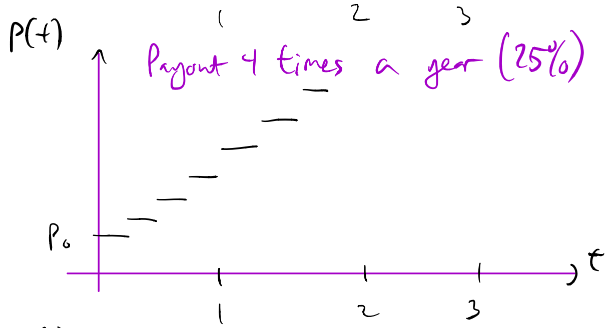
$\uparrow$ original $\quad$ $\uparrow$ gained



After 1 year:

$$P(1) = P_0\left(1+\tfrac{1}{2}\right)\left(1+\tfrac{1}{2}\right)$$

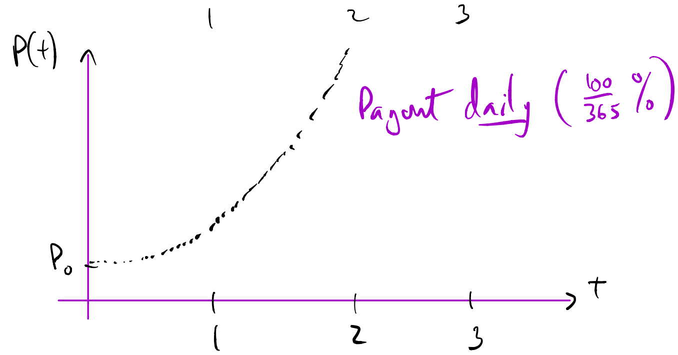
$$= P_0\left(1+\tfrac{1}{2}\right)^2 = 2.25P_0$$



After 1 year:

$$P(1) = P_0\left(1+\tfrac{1}{4}\right)\left(1+\tfrac{1}{4}\right)\left(1+\tfrac{1}{4}\right)\left(1+\tfrac{1}{4}\right)$$

$$= P_0\left(1+\tfrac{1}{4}\right)^4 \approx 2.441\,P_0$$

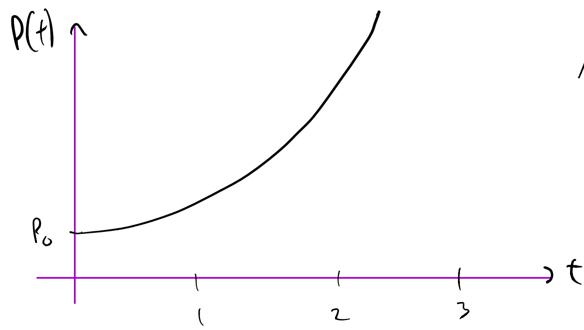


After 1 year:

$$P(1) = P_0\left(1+\tfrac{1}{365}\right)^{365}$$

$$\approx 2.7146\,P_0$$

In the limit, we say the interest is compounded continuously.



After 1 year:

$$P(1) = P_0 \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}_{\text{call this "e", } \approx 2.718281828\ldots}$$

e was first discovered in the early 1600s by Napier.

It arose several other times in the 1600s in different contexts.

In 1683, Jacob Bernoulli showed that $e < 3$.

[Note that by our above argument, $e > 2$, $e > 2.25$, ...]

Bernoulli: Define $\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$

Wed 10/3

Note that $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$

Plug in $x = \frac{1}{n}$:
$$\left(1+\frac{1}{n}\right)^n \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$
$$\leq 1 + 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$
$$= 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 3.$$

Similarly, we can define $\boxed{e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Derivative of e^x: Given $f(x) = e^x$,

$$f'(x) = \lim_{\Delta x\to 0}\frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x \to 0}\frac{e^{\Delta x}-1}{\Delta x}$$

$$= e^x \lim_{n\to 0}\frac{n}{\ln(n+1)} \cdot \frac{1/n}{1/n}$$

Let $n = e^{\Delta x} - 1$

$\iff n+1 = e^{\Delta x}$

$\iff \ln(n+1) = \Delta x$

$$= e^x \lim_{n\to 0} \frac{1}{\frac{1}{n}\ln(1+n)} = e^x \lim_{n\to 0} \frac{1}{\ln(1+n)^{1/n}}$$

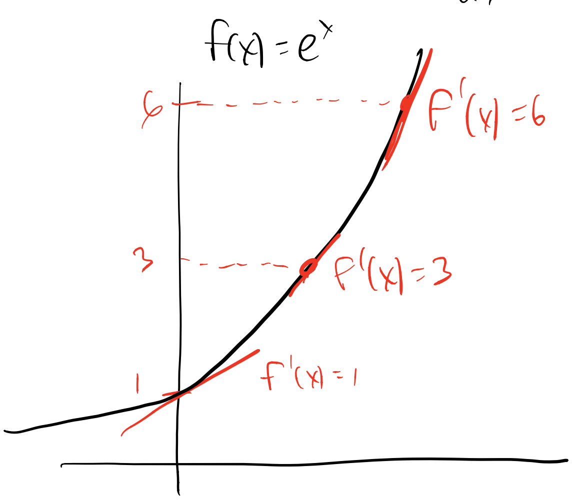
$$= e^x \lim_{n\to\infty} \frac{1}{\ln(1+1/n)^n} = e^x \cdot 1 = e^x$$

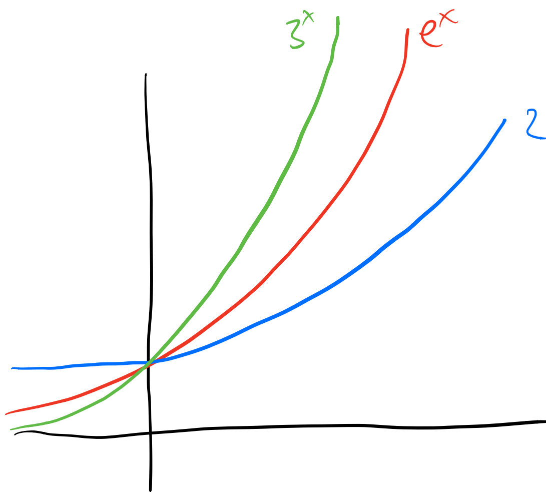
☆ Thus, $\boxed{\frac{d}{dx} e^x = e^x}$

By the chain rule, $\frac{d}{dx}\left(e^{kx}\right) = k e^{kx}$

$$\frac{d}{dx} e^{(x^2)} = e^{(x^2)} \cdot \frac{d}{dx} 2x = 2x e^{x^2}.$$

Fri 10/5

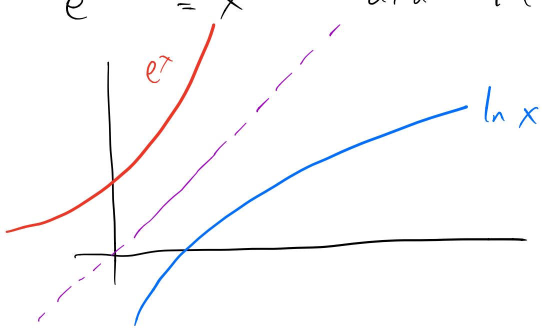




<u>Natural Logarithm</u>:

Recall that e^x and $\ln x$ are <u>inverse functions</u>, i.e.,

$$e^{\ln x} = x \quad \text{and} \quad \ln(e^x) = x$$



Derivatives of other exponential functions

Let $f(x) = 2^x = \left(e^{\ln 2}\right)^x = e^{(\ln 2)x}$

$f'(x) = (\ln 2)\, e^{(\ln 2)x} = (\ln 2)\, 2^x$

Similarly, $\frac{d}{dx} b^x = (\ln b)\, b^x$.

Derivative of natural log:

Suppose $y = \ln x$. Then $e^y = x$.

$$\frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(x)$$

$$e^y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Thus, $\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$.