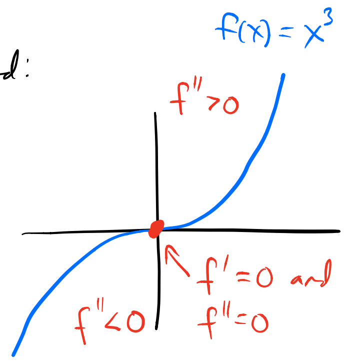
Week of Oct 8–12

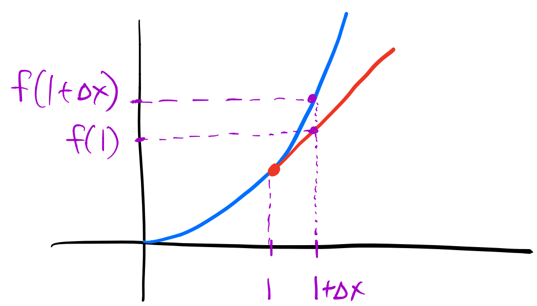
Summary of topics in differential calculus that we skipped:

- Inflection points: When $f'(x)=0$ but it's neither a local max or min

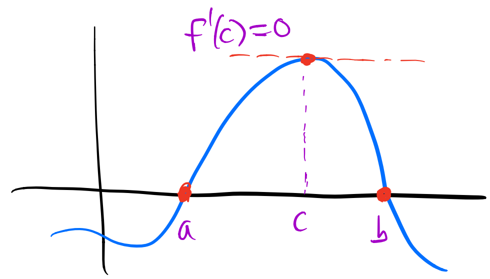


- Linear approximations:

  The tangent line to $f(x)$ at $(x_0, f(x_0))$ is a good approximation to $f(x)$ for $x \approx x_0$.

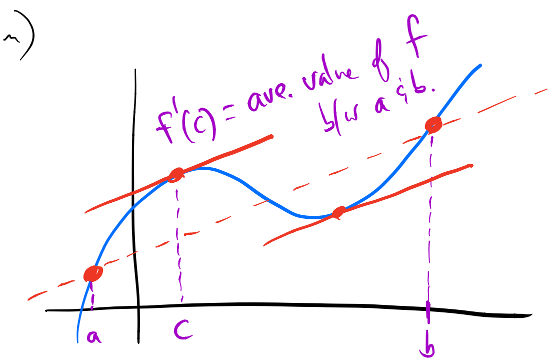


- Rolle's theorem: If $f(x)$ is continuous and $f(a)=f(b)$, then for some $c \in (a,b)$, $f'(c)=0$.



- Mean value theorem: (generalization of Rolle's thm)

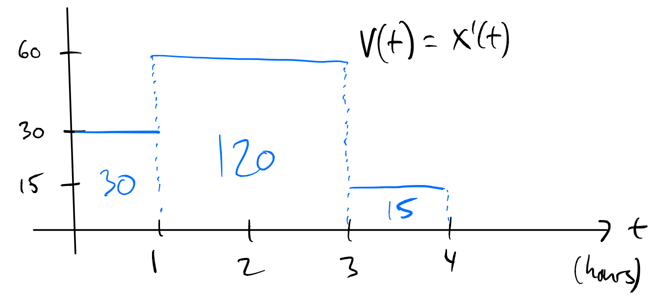
  If $f(x)$ is continuous, and $a<b$, then there is some $c \in (a,b)$ for which
  $$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



Note that Rolle's theorem is the special case of the MVT when $f(b)=f(a)$, and so $f'(c)=0$.

Next: We'll develop integral calculus and apply it to analyzing the domes of the Hagia Sophia, the Roman Pantheon, and the St. Louis Arch.

Motivating example: Consider a 4-hr road trip, where the velocity you traveled is the following:



☆ Question: How far did you travel?

There are two ways to answer this:

Method #1: "area under curve"

$0 \le t \le 2$: $\left(30\,\frac{mi}{hr}\right)(1\ hr) = 30\ mi$

$1 \le t \le 3$: $\left(60\,\frac{mi}{hr}\right)(2\ hr) = 120\ mi$

$3 \le t \le 4$: $\left(15\,\frac{mi}{hr}\right)(1\ hr) = 15\ mi$

total distance: $30 + 120 + 15 = 165$ mi

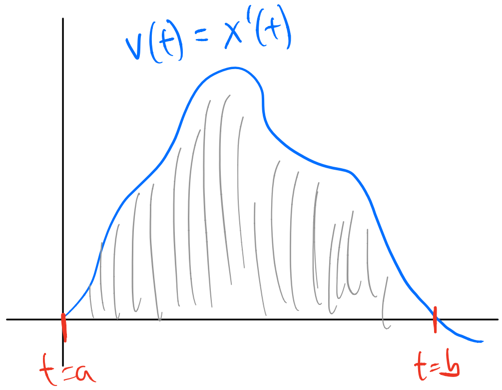
Method #2: "odometer"

Check your odometer after & before the trip. Subtract these values:

$$x(4) - x(0) = \boxed{5|2|0|0|0} - \boxed{5|0|3|5} = 165\ mi.$$

☆ This is half of the "Fundamental theorem of calculus."

It works more generally, not just for piecewise functions.



velocity is the derivative of distance.

Total distance is:

- Area under the curve of $x'(t)$
- $x(b) - x(a)$.

Big idea: Function $f(x) \xrightarrow{\text{take derivative}} f'(x)$ "rate of change"

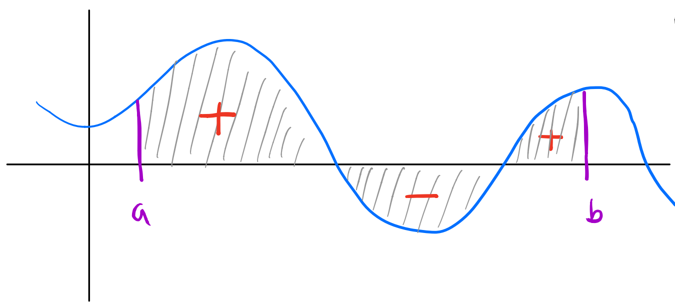
$f(x) \xleftarrow{\text{area under curve}} f'(x)$

Week of Oct 15-19

★ Key concept: "net area", or "signed area" from a to b:

Notation (to be explained later):

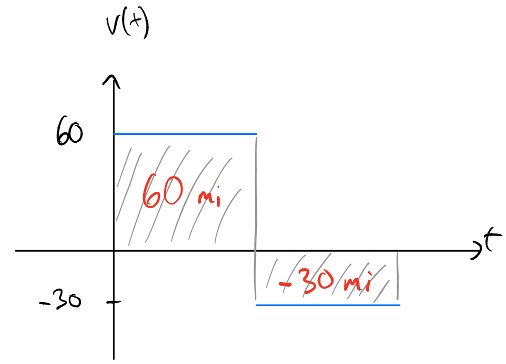
$$\int_a^b f(x)\,dx$$

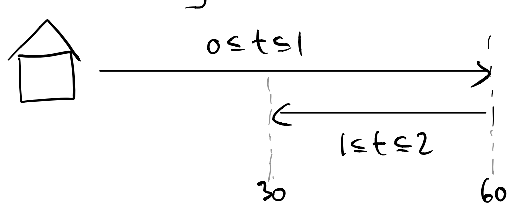


Example: Consider the following road trip:

$0 \le t \le 1$: Traveling away from home at 60 mph

$1 \le t \le 2$: Traveling towards home at 30 mph.





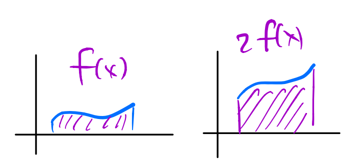
Properties:

① $\int_a^a f(x)\,dx = 0$

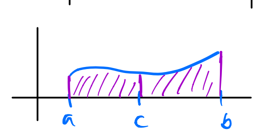
② $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

③ $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

④ $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$



⑤ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



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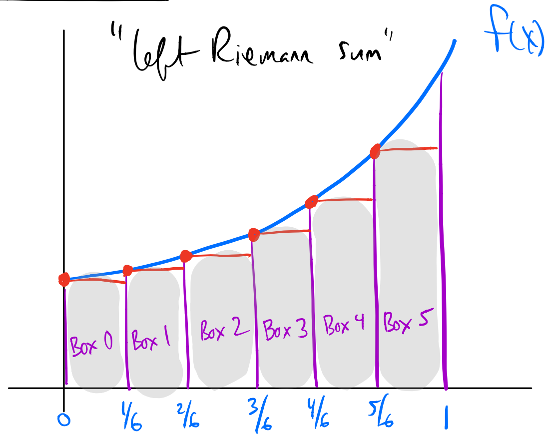
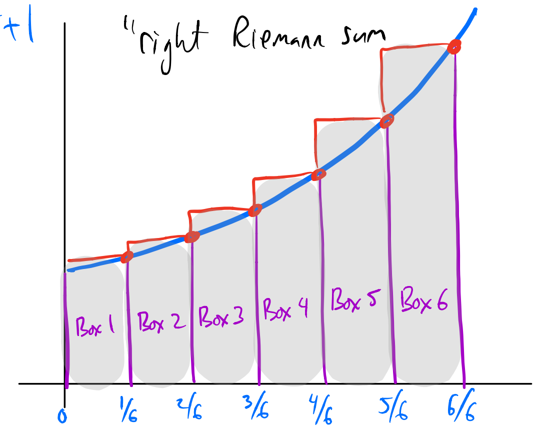
Riemann sums "approximate area under the curve"

$f(x) = x^2 + 1$



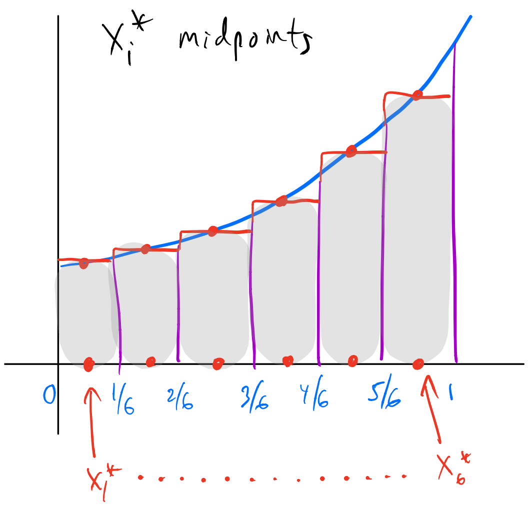


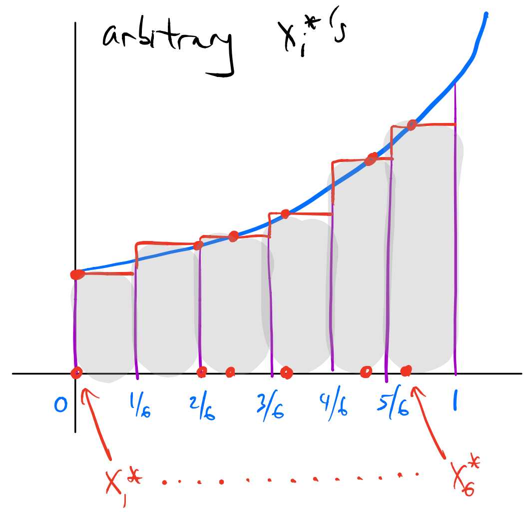
Let $\Delta x = 1/6$

Left Riemann sum: Area $= f(0)\,\Delta x + f(1)\,\Delta x + f(2)\,\Delta x + f(3)\,\Delta x + f(4)\,\Delta x + f(5)\,\Delta x$

Right Riemann sum: Area $= f(1)\,\Delta x + f(2)\,\Delta x + f(3)\,\Delta x + f(4)\,\Delta x + f(5)\,\Delta x + f(6)\,\Delta x.$

Alternatively, the midpoint, or any other point can be chosen as height.





Regardless of which Riemann sum we chose,

$$\text{Area} = \lim_{\Delta x \to 0} \left(\sum_{i=1}^{n} f(x_i^*)\, \Delta x \right) = \int_a^b f(x)\, dx$$

$\int_a^b$ $f(x)$ dx

Recall "sigma notation": $\displaystyle\sum_{k=1}^{6} k = 1+2+3+4+5+6 = \sum_{i=1}^{6} i.$

"dummy variable"

Properties: $\displaystyle\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

$$\sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$$

Identities: $\displaystyle\sum_{k=1}^{n} k = 1+2+3+\cdots+(n-1)+n = \frac{n(n+1)}{2}$

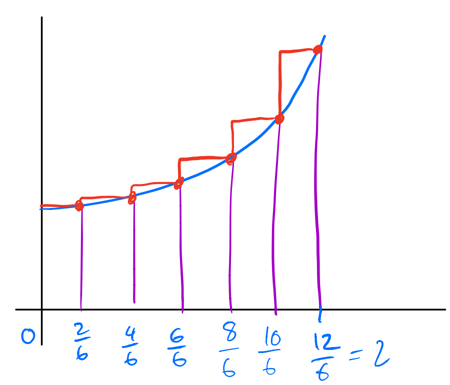
$$\sum_{k=1}^{n} k^2 = 1+4+9+\cdots+(n-1)^2+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1+8+27+\cdots+(n-1)^3+n^3 = \frac{n^2(n+1)^2}{4}$$

Riemann sum example: Compute $\displaystyle\int_0^2 (x^2+1)\,dx$. Let $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Subintervals $\left[0,\frac{2}{n}\right], \left[\frac{2}{n},\frac{4}{n}\right], \ldots, \left[\frac{2(n-1)}{n}, 2\right]$

Right endpoint: $x_i^* = \frac{2i}{n}$



$$\text{Area} = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n}$$

$$= \sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^2+1\right)\cdot\frac{2}{n} = \sum_{i=1}^{n}\left(\frac{8i^2}{n^3}+\frac{2}{n}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{8i^2}{n^3}+\frac{2}{n}\right) = \underbrace{\lim_{n\to\infty}\sum_{i=1}^{n}\frac{8i^2}{n^3}}_{(*)} + \underbrace{\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}}_{(**)}$$

$\boxed{6}$

$$(*) \quad \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{8}{n^3} \cdot \left[\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty} \frac{8}{6} \cdot \frac{(n+1)(2n+1)}{n^2} = \boxed{\frac{4}{3}}$$

$$(**) \quad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} = \lim_{n\to\infty} \frac{2}{n} \cdot \sum_{i=1}^{n} 1 = \frac{2}{n} \cdot n = \boxed{2}$$

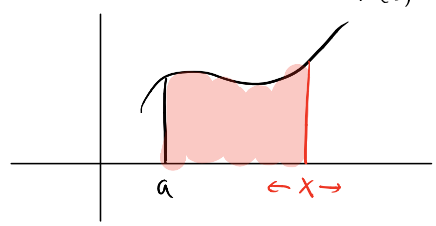
Thus, $\int_0^2 (x^2+1)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{4}{3} + 2 = \boxed{\frac{10}{3}}$

Week of Oct 22-26

Area function Fix $f(x)$ and a real number, a.

Define $A(x) = \int_a^x f(t)\,dt$

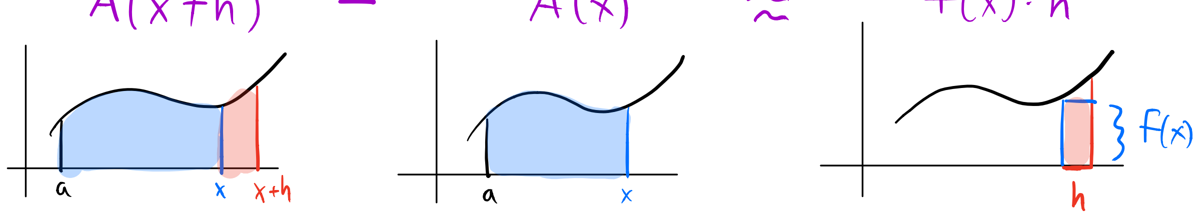
= "area under the curve of f, from a to x."



Clearly, $A(a) = \int_a^a f(t)\,dt = 0$.

Remark:

$$A(x+h) \quad - \quad A(x) \quad \approx \quad f(x)\cdot h$$



Note that $A(x+h) - A(x) \approx f(x)\cdot h$

$$\Rightarrow \frac{A(x+h)-A(x)}{h} \approx f(x)$$

Take $\lim_{h\to 0}$ of both sides: $\boxed{A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x)}$

Big idea: If $A(x)$ is the area function of $f(x)$, then $\frac{d}{dx}(A(x)) = f(x)$.

In other words: "the derivative & area functions are inverse functions."

This is the Fundamental Theorem of Calculus, Part 1.

If f is continuous on $[a,b]$ and differentiable on (a,b) then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

We say that $A(x)$ is an antiderivative of $f(x)$.

Proposition: If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ (i.e., $F'(x) = G'(x) = f(x)$), then $F(x) - G(x) = C$, for some constant. (Because $F' - G' = 0 \Rightarrow F - G = C$).

Consider a function $f(x)$. We know $A(x)$ is an antiderivative.
Let $F(x)$ be any other antiderivative.

$$\text{Then } F(x) = A(x) + C \Rightarrow F(b) - F(a) = (A(b) + C) - (A(a) + c) \quad [\text{Recall: } A(a) = 0.]$$
$$= A(b) = \int_a^b f(x)\,dx.$$

This is the Fundamental Theorem of Calculus, Part 2:

If f is continuous on $[a,b]$ and F is any antiderivative of f, then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Before we do examples, let's practice anti-derivatives

Notation: If $F'(x) = f(x)$, then we write $\int f(x)\,dx = F(x) + C$.

We call this an indefinite integral.

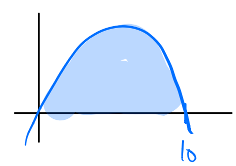
A definite integral has the form $\int_a^b f(x)\,dx$.

Practice with antiderivatives (indefinite integrals)

- $\int 1\,dx = x + C$
- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad n \neq -1$
- $\int x^2\,dx = \frac{1}{3}x^3 + C$
- $\int (3x^2 - 4)\,dx = x^3 - 4x + C$
- $\int e^x\,dx = e^x + C$
- $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \sin kx\,dx = \frac{-1}{k}\cos kx + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \cos kx\,dx = \frac{1}{k}\sin kx + C$
- $\int \frac{1}{x}\,dx = \ln x + C$
- $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \dfrac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C.$

Practice with definite integrals.

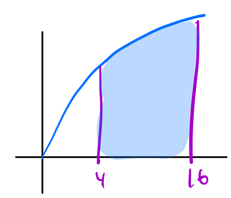
- $\int_0^{10} 60x - 6x^2\,dx = (30x^2 - 2x^3)\Big|_{x=0}^{x=10}$



$= (30(10^2) - 2(10^3)) - (30(0^2) - 20(0^2))$

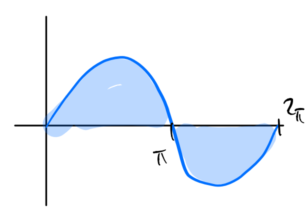
$= 3000 - 2000 = \boxed{1000}$

- $\int_4^{16} 3\sqrt{x}\,dx = \int_4^{16} 3x^{1/2}\,dx = \dfrac{3x^{3/2}}{3/2}\Big|_4^{16}$



$= 2x^{3/2}\Big|_4^{16} = 2\sqrt{x^3}\Big|_4^{16} = 2\sqrt{16^3} - 2\sqrt{4^3} = 2\cdot 64 - 2\cdot 8 = \boxed{112}$

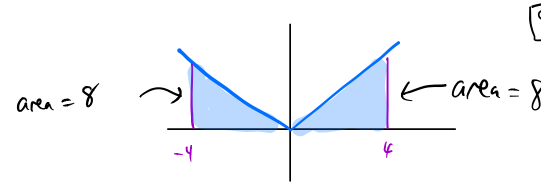
- $\int_0^{2\pi} 3\sin x\,dx = -3\cos x\Big|_0^{2\pi}$



$= (-3\cos 2\pi - -3\cos 0)$

$= -3\cdot 1 + 3\cdot 1 = \boxed{0}$

- $\int_{-4}^{4} |x| \, dx = 16$ (see graph)



OR $= \int_{-4}^{0} -x \, dx + \int_{0}^{4} x \, dx$

$= \quad 8 \quad + \quad 8$ (exercise)

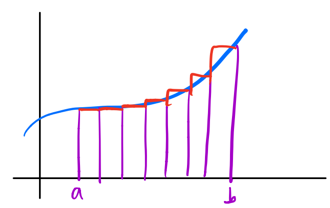
Application: Average value of a function.

Consider 3 football teams:

Clemson: 8 wins

NCSU: 5 wins

USC: 3 wins



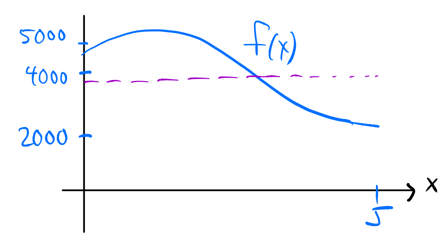
The average # wins $= \frac{8+5+3}{3} = \frac{16}{3} = 5.33\ldots$

Now, consider a continuous quantity $f(x)$.

The average value of $f(x)$ on $a \leq x \leq b$ is $\frac{1}{b-a} \int_a^b f(x) \, dx$.

Ex: Suppose the elevation on a 5-mile hike is $f(x) = 60x^3 - 650x^2 + 1200x + 4500$ ft.

Find the average elevation.



Ans: $\frac{1}{5-0} \int_0^5 60x^3 - 650x^2 + 1200x + 4500 \, dx$

$= \frac{1}{5} \left(15x^4 - \frac{650}{3} x^3 + 600x^2 + 4500x \right) \Big|_0^5$

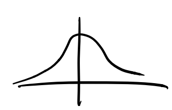
$= \frac{1}{5} \left(15(5^4) - \frac{650}{3}(5^3) + 600(5^2) + 4500(5) \right) \approx \boxed{3958.3 \text{ ft}}$

Derivatives vs. integrals

Derivatives have formulas: product, quotient, chain rule.

So we can always take the derivative of e.g., $\sin\left(e^{(x^2 + \cos x)}\right)$.

Integrals have no such formulas. For example, $\int e^{-x^2} dx$ has no "closed form".



Computing integrals is more of an "art".

Recall the chain rule: $\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$ [where $F' = f$.]

Now integrate both sides: $\int f(g(x))\, g'(x)\, dx = F(g(x)) + C$

$\int f(u)\, du = F(u) + C.$

Let $u = g(x)$

so $\frac{du}{dx} = g'(x)$

$\Rightarrow du = g'(x)\, dx$

Examples

① $\int 2(2x+1)^{30}\, dx.$ Let $u = 2x+1$ $\frac{du}{dx} = 2 \Rightarrow du = 2\, dx$

$= \int u^{30}\, du = \frac{1}{31} u^{31} + C = \frac{1}{31}(2x+1)^{31}\, dx$

② $\int 3 e^{10x}\, dx.$ Let $u = 10x$, $\frac{du}{dx} = 10 \Rightarrow du = 10\, dx \Rightarrow \frac{3}{10}\, du = 3\, dx$

$= \int \frac{3}{10} e^{u}\, du = \frac{3}{10} e^{u} + C = \frac{3}{10} e^{10x} + C.$

③ $\int x^4 (x^5+6)^9\, dx.$ Let $u = x^5+6 \Rightarrow \frac{du}{dx} = 5x^4 \Rightarrow du = 5x^4\, dx \Rightarrow \frac{1}{5} du = x^4 dx$

$= \int u^9 \cdot \frac{1}{5}\, du = \frac{1}{5}\int u^9\, du = \frac{1}{5}\, \frac{u^{10}}{10} + C = \frac{u^{10}}{50} + C = \frac{(x^5+6)^{10}}{50} + C.$

④ $\int (\cos x)^3 \sin x\, dx.$ Let $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow -du = \sin x\, dx$

$= \int -u^3\, dx = \frac{-u^4}{4} + C = \frac{-(\cos x)^4}{4} + C.$

Week of Oct 29 – Nov 2.

⑤ $\int \frac{x\, dx}{\sqrt{1+x^2}}.$ Let $u = 1+x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2} du = x\, dx$

$= \int \frac{1}{2} u^{-1/2}\, du = \frac{1}{2}\, \frac{u^{1/2}}{1/2} + C = u^{1/2} + C = \sqrt{1+x^2} + C.$

⑥ $\int 3x^2 \sqrt{x^3+1}\, dx$ Let $u = x^3+1 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2\, dx$

$= \int \sqrt{u}\, du = \int u^{1/2}\, du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3} u^{3/2} + C = \frac{2}{3}(x^3+1)^{3/2} + C = \frac{2}{3}\sqrt{(x^3+1)^3} + C.$

⑦ $\int \frac{\sin t}{2-\cos t}\, dt.$ Let $u = 2-\cos t \Rightarrow \frac{du}{dt} = -\sin t \Rightarrow du = -\sin t\, dt$

$= \int -\frac{1}{u}\, du = -\ln|u| + C = -\ln|2-\cos t| + C.$