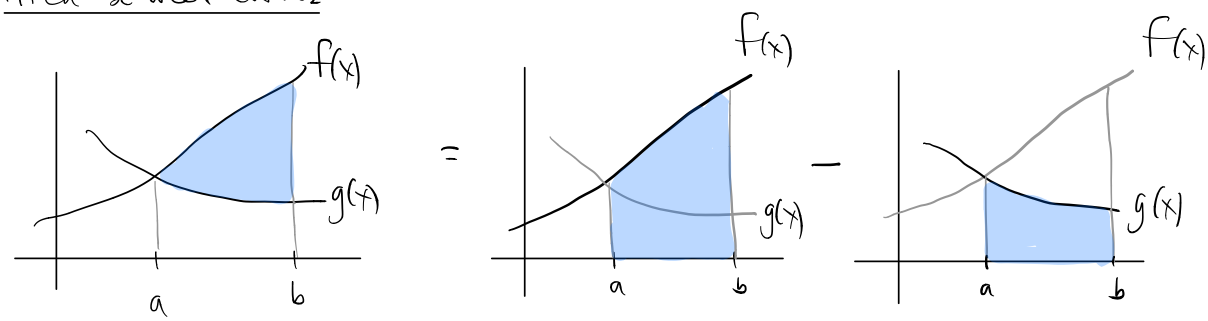
Week of Oct 29 – Nov 2

Area between curves



$$\int_a^b \left[f(x) - g(x)\right] dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

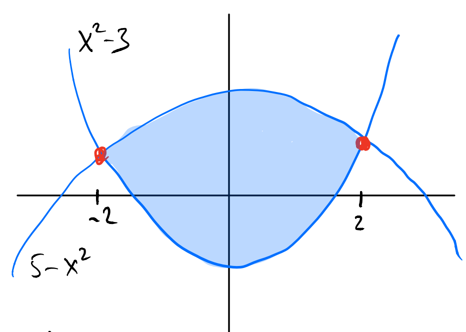
This even works if the region is below or straddles the x-axis.

Example 1: Find the area between the curves of $f(x) = 5 - x^2$ and $g(x) = x^2 - 3$

First, find points of intersection:

$5 - x^2 = x^2 - 3$

$8 = 2x^2 \Rightarrow x = \pm 2$



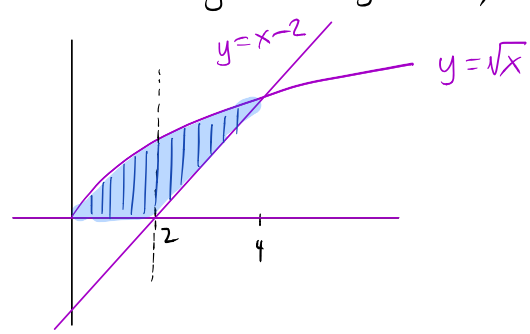
$\text{Area} = \int_{-2}^{2} (5 - x^2) - (x^2 - 3)\, dx$

$= \int_{-2}^{2} 8 - 2x^2\, dx = 8x - \frac{2}{3}x^3 \Big|_{-2}^{2} = \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right) = 32 - \frac{32}{3} = \boxed{\frac{64}{3}}$

Example 2: Find the area between the curves: $y = \sqrt{x}$, $y = x - 2$, and the x-axis.

$\text{Area} = \int_0^2 (\sqrt{x} - 0)\, dx + \int_2^4 \sqrt{x} - x\, dx$

Note that we'll need to break this into two integrals.



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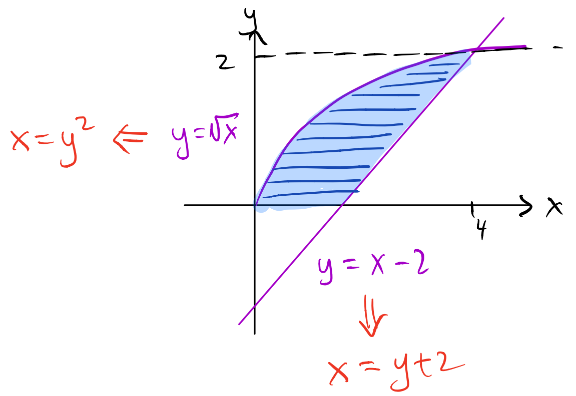
$$\text{Area} = \int_0^2 (\sqrt{x} - 0)\, dx + \int_2^4 \sqrt{x} - (x-2)\, dx$$

$$= \frac{2}{3} x^{3/2} \Big|_0^2 + \left(\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right) \Big|_2^4$$

$$= \cancel{\frac{2}{3}\sqrt{8}} + \left[\left(\frac{16}{3} - \cancel{8} + \cancel{8} \right) - \left(\cancel{\frac{2}{3}\sqrt{8}} - 2 + 4 \right) \right] = \frac{16}{3} + \frac{6}{3} - \frac{12}{3} = \boxed{\frac{10}{3}}$$

Week of Nov 5-9

Another method: Integrate w.r.t. y.



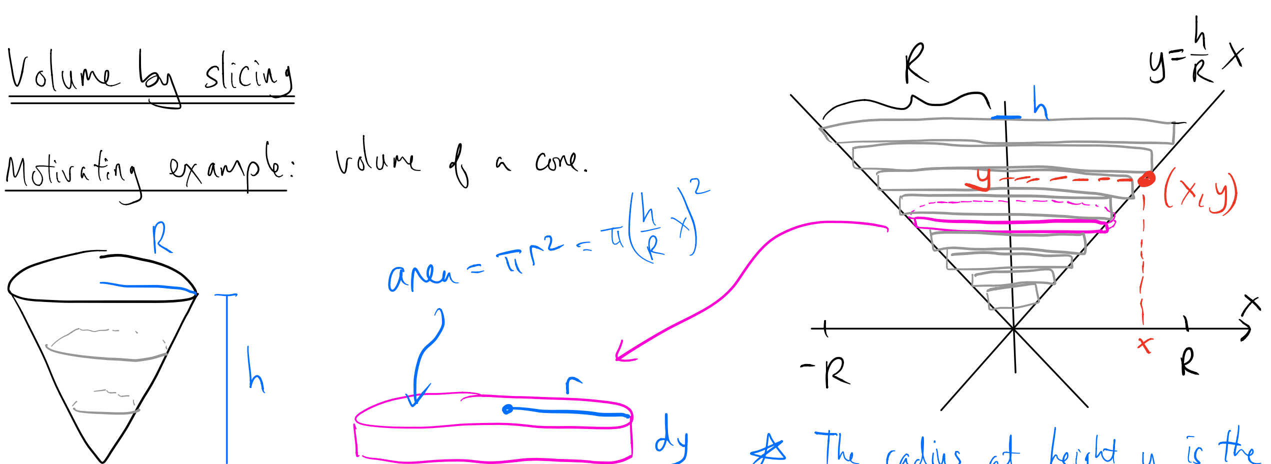
$$\text{Area} = \int_0^2 (y+2) - y^2 \; dy$$

$$= \left(\frac{y^2}{2} + 2y - \frac{y^3}{3} \right) \Big|_0^2$$

$$= \left[\left(2 + 4 - \frac{8}{3} \right) - (0 + 0 - 0) \right] = \frac{18}{3} - \frac{8}{3} = \boxed{\frac{10}{3}}$$ (much easier!)

Volume by slicing

Motivating example: volume of a cone.



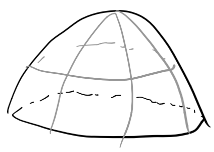
☆ The radius at height y is the x-value: $y = \frac{h}{R} x \Rightarrow \boxed{x = \frac{R}{h} y}$

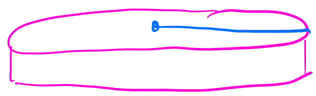
$$\text{Volume of slice} = \pi r^2 = \pi \left(\frac{h}{R} y \right)^2 dy$$

$$\text{Volume of cylinder} = \int_0^h \text{"vol of slice @ height } y\text{"}$$

$$= \int_0^h \pi\left(\frac{R}{h}y\right)^2 dy = \int_0^h \frac{\pi R^2}{h^2} y^2\, dy = \frac{\pi R^2}{h^2}\frac{y^3}{3}\Big|_0^h = \frac{\pi R^2 h^3}{3h^2} = \boxed{\frac{1}{3}\pi R^2 h}$$

<u>Example</u>: Volume of a hemisphere.

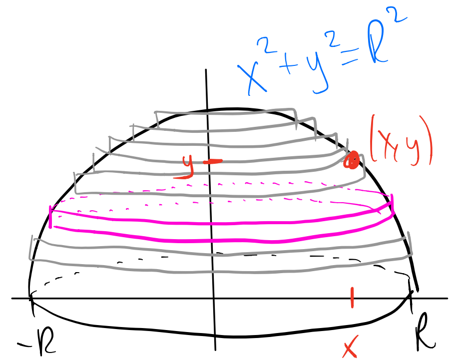




$r = x\text{-value} = \sqrt{R^2 - y^2}$

dy

$\text{Vol} = \pi r^2\, dy = \pi(R^2 - y^2)\,dy$



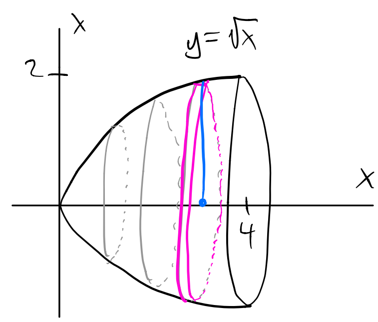
$$\text{Vol. of cylinder} = \int_0^R \text{"vol. of slice at height } y\text{"}$$

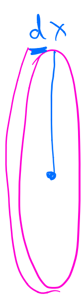
$$= \int_0^R \pi(R^2-y^2)\,dy = \int_0^R \pi R^2 - \pi y^2\, dy$$

$$= \left(\pi R^2 y - \pi \frac{1}{3}y^3\right)\Big|_0^R = \pi R^3 - \frac{1}{3}\pi R^3 = \boxed{\frac{2}{3}\pi R^3}$$

Volume of other "<u>solids of revolution</u>".

<u>Ex</u>: Consider the solid formed by revolving the region under $y=\sqrt{x}$ from $x=0$ to 4 around the x-axis. Find its volume.





$r = \text{"y-value"} = \sqrt{x}$

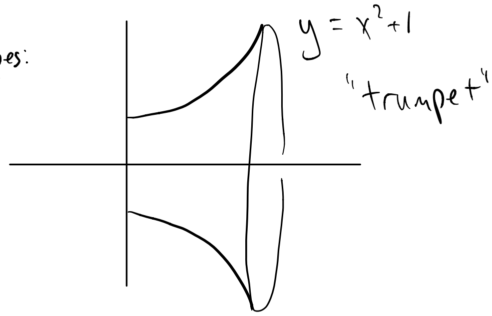
$\text{Vol} = \pi r^2\, dx$
$= \pi(\sqrt{x})^2\, dx$
$= \pi x\, dx$

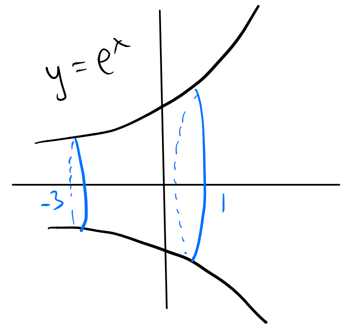
$$\text{Vol} = \int_0^4 \pi x\, dx = \pi \frac{x^2}{2}\Big|_0^4 = \boxed{8\pi}$$

Sometimes this method is called the "disk method".

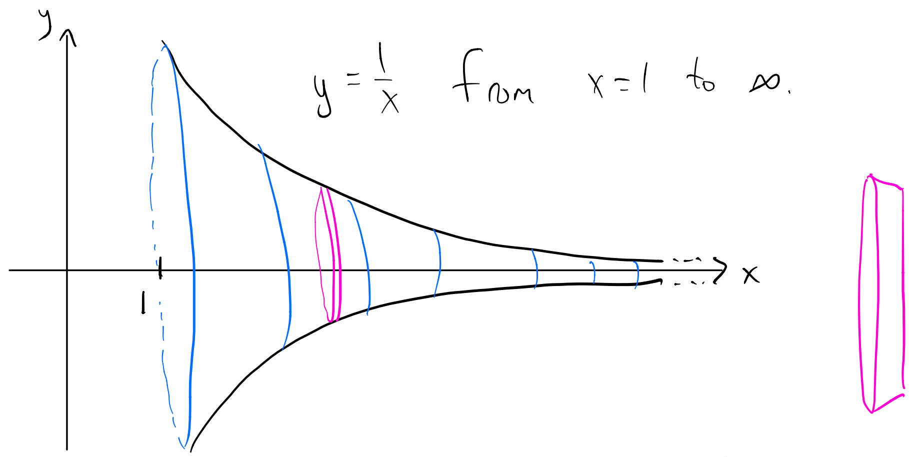
$\boxed{4}$

we can do other shapes:





"Gabriel's horn"

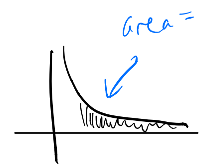


First we need to see what are called "improper integrals", i.e., integrating over an asymptote, or where a limit is ∞.

Big idea: "treat ∞ as any ordinary number."

Examples: $\int_1^\infty \frac{1}{x}\,dx = \ln x \Big|_1^\infty = \ln\infty - \ln 1 = \infty$

$$\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty} - \left(-\frac{1}{1}\right) = 0+1 = 1$$

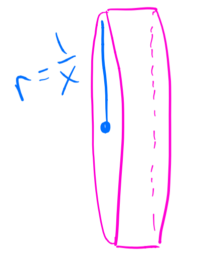


Back to Gabriel's horn:

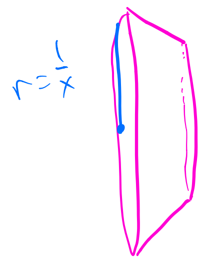
$$\text{Volume} = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \int_1^\infty \pi\frac{1}{x^2}\,dx = \pi\int_1^\infty \frac{1}{x^2}\,dx = \pi.$$

We can also compute the <u>surface area</u>

Technically, we don't know how to do this, but we can show it's infinite



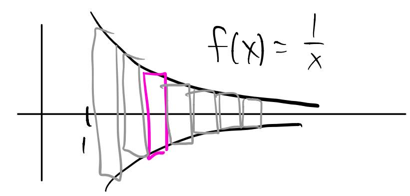
Surface area $= 2\pi r = \frac{2\pi}{x}$



Surface area $= 2\pi r \underbrace{\sqrt{1 + (f'(x))^2}}_{\text{accounts for slant}}$

$$= \frac{2\pi}{x}\sqrt{1 + \frac{1}{x^4}}$$

$$< \frac{2\pi}{x}$$



Thus, the surface area is <u>bigger</u> than

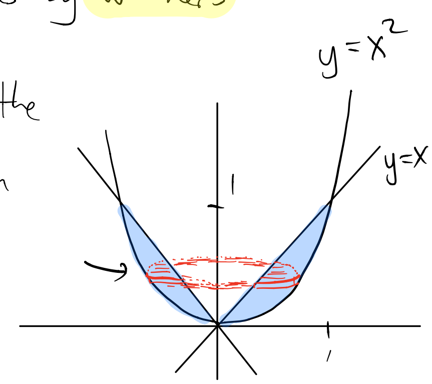
$$\int_1^\infty 2\pi r \, dx = \int_1^\infty \frac{2\pi}{x}\, dx$$

$$= \frac{2}{\pi}\int_1^\infty \frac{1}{x}\, dx = \infty$$

☆ So, Gabriel's horn has <u>finite volume</u> but <u>infinite surface area</u>!

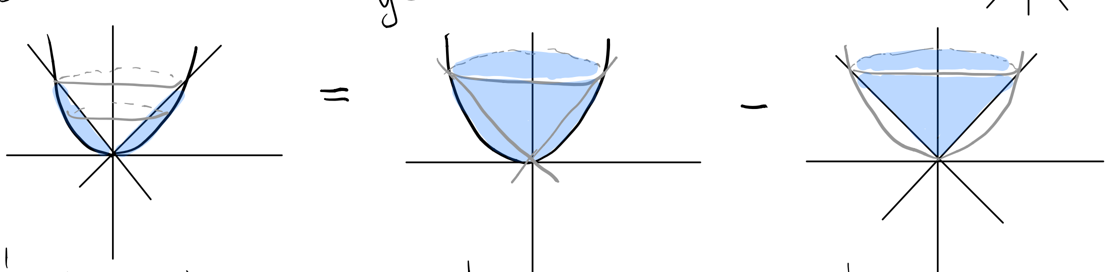
The following method is sometimes called "volumes by washers"

<u>Example</u>: Consider the region formed by rotating the area between the curves $y = x$ and $y = x^2$ from $x = 0$ to $x = 1$ around the y-axis.



Key idea:

$y = x^2 \Rightarrow x = \sqrt{y}$



$$\int_0^1 \pi\left(\sqrt{y}^2 - y^2\right) dy = \int_0^1 \pi \sqrt{y}^2\, dy - \int_0^1 \pi y^2\, dy$$

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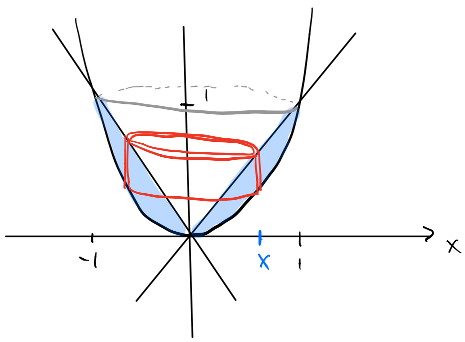
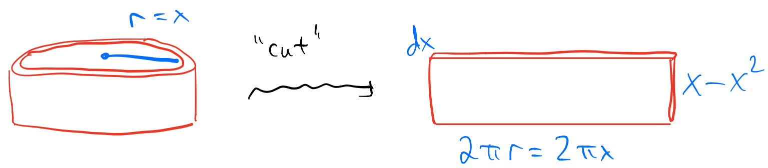
$$= \pi \frac{y^2}{2}\Big|_0^1 - \pi \frac{y^3}{3}\Big|_0^1 = \frac{\pi}{2} - \frac{\pi}{3} = \boxed{\frac{\pi}{6}}$$

Week of Nov 12–16

We can do the previous problem using a different method called **shells**.





$$\text{Vol} = (\text{area})(\text{height})(\text{depth})$$
$$= 2\pi x(x - x^2)\,dx$$

$$\text{Vol} = \int_0^1 \text{"vol of shell of radius } x\text{"}$$

$$= \int_0^1 2\pi x(x - x^2)\,dx = \int_0^1 2\pi(x^2 - x^3)\,dx = 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{\frac{\pi}{6}}$$