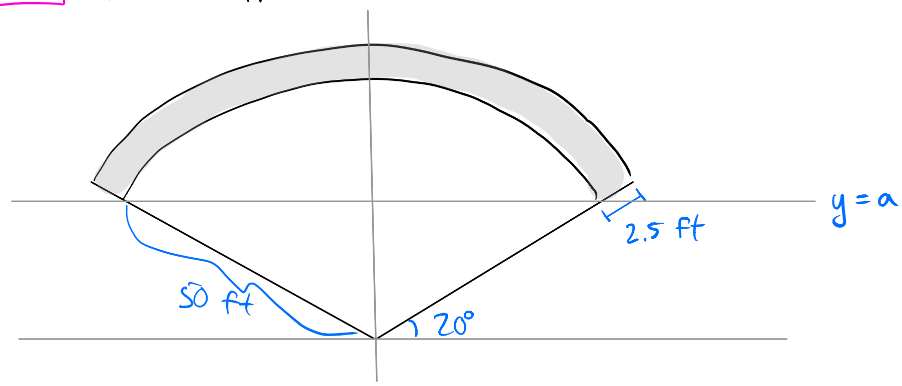
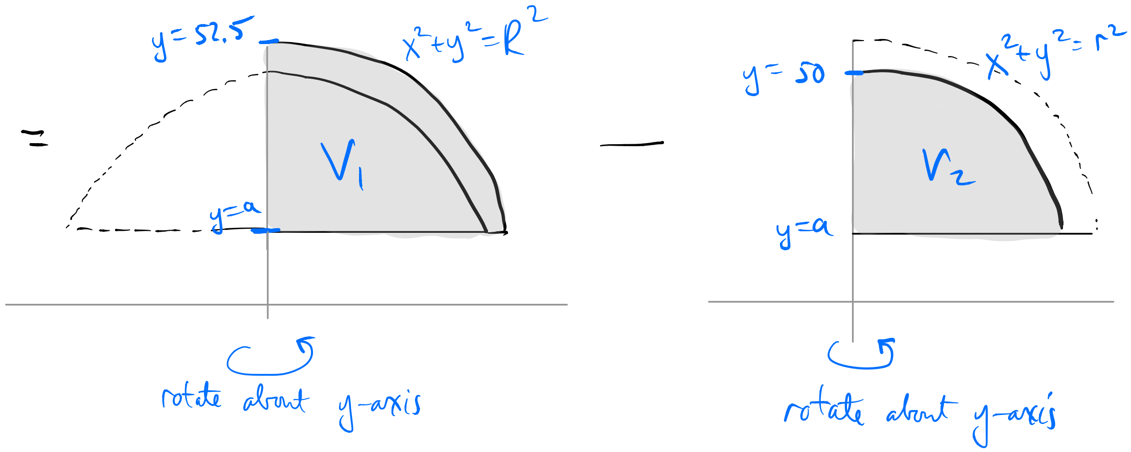
Week of Nov 12–16 (also see ppt slides)

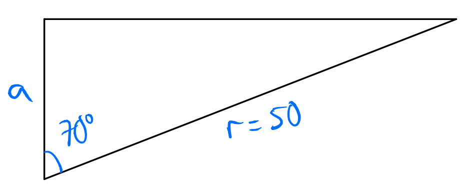
Hagia Sophia

Cross section:





Find a:



$\cos 70^\circ = \frac{50}{a}$

$\Rightarrow a = 50 \cos 70^\circ \approx 17.$

So $R = 52.5$, $r = 50$, $a = 17$.

$$V_1 = \int_a^R \pi\left(\sqrt{R^2 - x^2}\right)^2 dx = \int_a^R \pi(R^2 - x^2)\, dx = \pi\left[R^2 x - \tfrac{1}{3}x^3\right]\Big|_a^R$$

$$= \pi\left[\left(R^3 - \tfrac{1}{3}R^3\right) - \left(R^2 a - \tfrac{1}{3}a^3\right)\right] = \pi\left[\tfrac{2}{3}R^3 - R^2 a + \tfrac{1}{3}a^3\right] \approx \pi(13{,}135.42)$$

Similarly, $V_2 = \int_a^r \pi\left(\sqrt{r^2 - x^2}\right)^2 dx = \dots = \pi\left[\tfrac{2}{3}r^3 - r^2 a + \tfrac{1}{3}a^3\right] \approx \pi(4356.25)$.

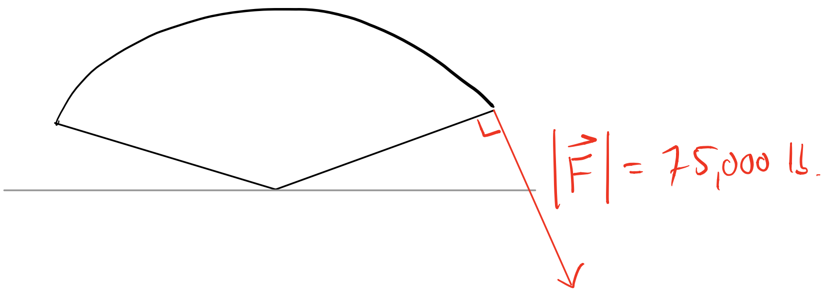
So $V = V_1 - V_2 = 27{,}581\ \text{ft}^3 \approx 28{,}000\ \text{ft}^3$

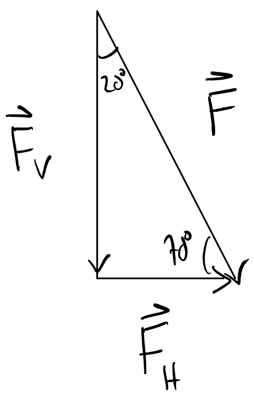
2

Concrete weighs $\approx 110 \text{ lb}/\text{ft}^3 \Rightarrow$ total weight is $\approx 3{,}000{,}000$ lbs.

There are 40 supporting ribs $\Rightarrow \dfrac{3{,}000{,}000}{40} = 75{,}000$ lbs per buttress.

- Calculate the outward force at each buttress:



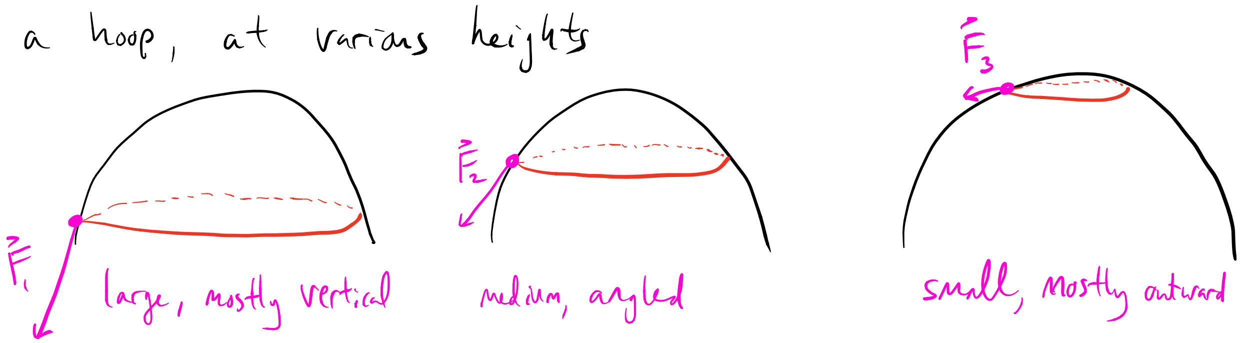


$$|\vec{F}_H| = |\vec{F}| \cdot \cos 70^\circ$$

$$= (75{,}000)(0.342) \approx \boxed{27{,}000 \text{ lbs}}$$

Def:
- Vertical cross-sections of a dome are called meridians
- Horizontal cross-sections of a dome are called hoops.

Compare the force (magnitude & direction) of the dome on a hoop, at various heights



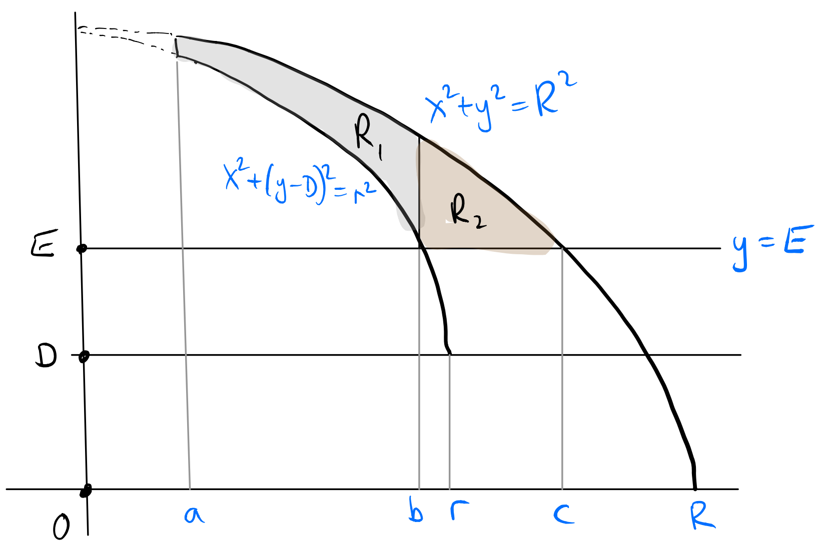
The hoop stress is the outward force.

★ First principle of structural architecture: Unless the shell can resist the "hoop stress," the shell will expand along the hoops ; cracks will develop along meridians.

Week of Nov. 19–23 : No class

Week of Nov. 26–30 : (Also see ppt slides)

Roman Pantheon



$r = 71$ feet

$R = 92$

$D = 16$

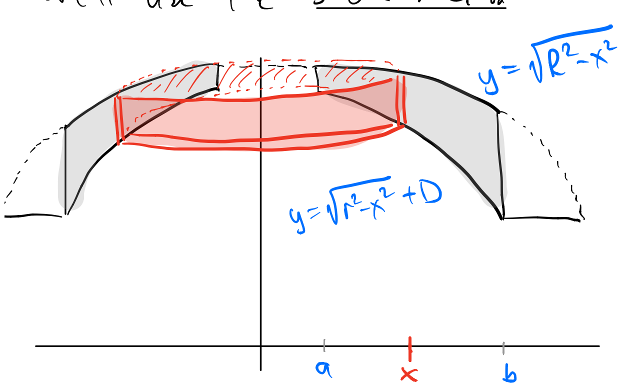
$E = 48$

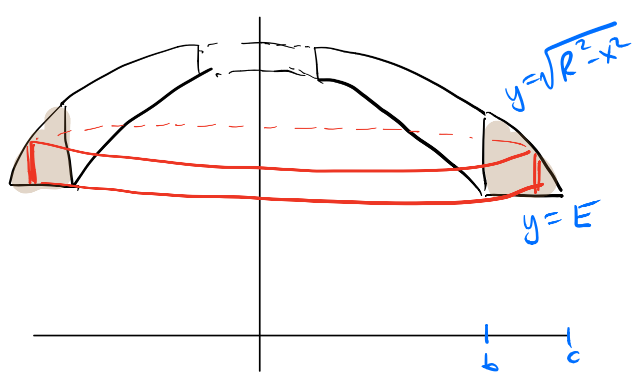
$a = 12$

$b = 64$

$c = 78$

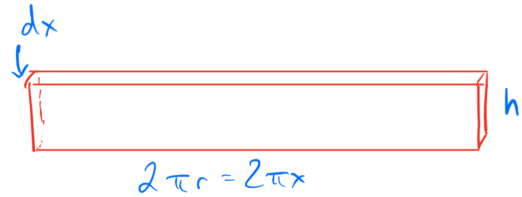
We'll use the shell method





$\boxed{4}$

Recall: Vol of a shell:



$$= (\text{base})(\text{height})(\text{thickness})$$

circumference dx

$$\text{Vol } 1 = \int_a^b 2\pi x\left[\sqrt{R^2-x^2} - \left(\sqrt{r^2-x^2} + D\right)\right] dx$$

$$= \int_a^b 2\pi x\sqrt{R^2-x^2}\, dx - \int_a^b 2\pi x\sqrt{r^2-x^2}\, dx - \int_a^b 2\pi x D\, dx$$

let $u = R^2 - x^2 \Rightarrow du = -2x\, dx$

$$= \int_a^b -\pi\sqrt{u}\, du + \int_a^b \pi\sqrt{u}\, du - \int_a^b 2\pi D x$$

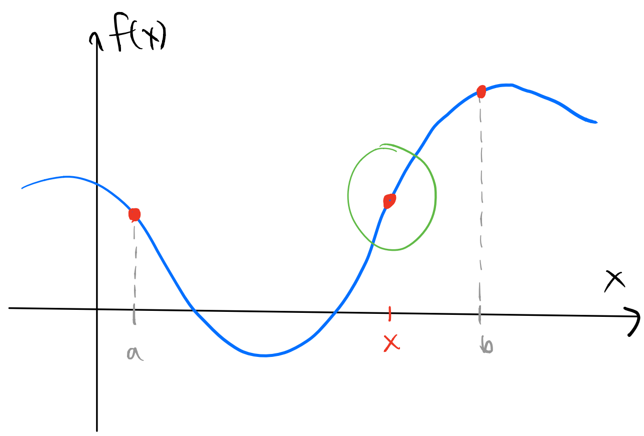
$$= -\frac{2}{3}\pi u^{3/2}\Big|_{x=a}^{x=b} + \frac{2}{3}\pi u^{3/2}\Big|_{x=a}^{x=b} - \pi D x^2\Big|_a^b$$

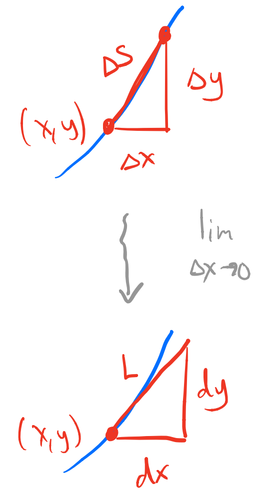
$$= -\frac{2}{3}\pi\left(R^2-x^2\right)^{3/2}\Big|_a^b + \frac{2}{3}\pi\left(r^2-x^2\right)\Big|_a^b - \pi D\left(b^2-a^2\right)$$

$$= -656{,}875 + 984{,}818 - 198{,}694 = \boxed{129{,}299 \text{ ft}^3}$$

Arc length

Goal: Find the length of a curve $y=f(x)$, from $x=a$ to $x=b$



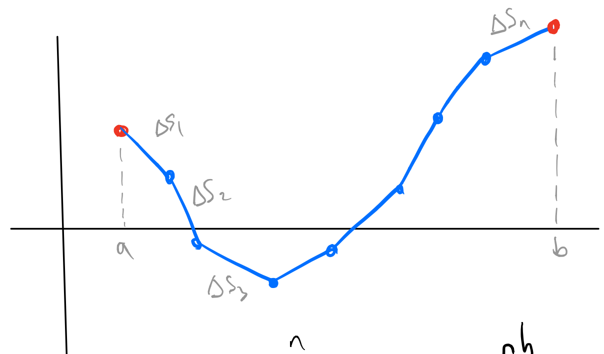


Think of arc length of the limit of the sum of small pieces, as $\Delta S \to 0$

$$L = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2}$$

$$= \sqrt{1 + (f'(x))^2}\ dx$$



$$\text{Arc length} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \Delta S_i = \int_a^b ds = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \boxed{\int_a^b \sqrt{1 + (f'(x))^2}\ dx}$$

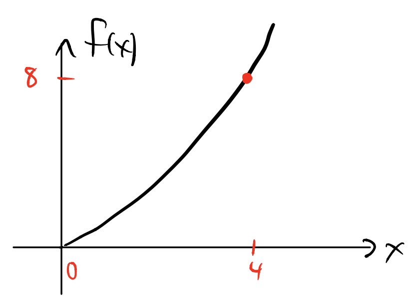
Example: Find the arc length of $y = \sqrt{x^3} = x^{3/2}$ from $x=0$ to $x=4$

$$\int_0^4 \sqrt{1 + \left(\frac{d}{dx} x^{3/2}\right)^2}\ dx = \int_0^4 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2}\ dx$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}x}\ dx$$

let $u = 1 + \frac{9}{4}x$

$du = \frac{9}{4}dx$



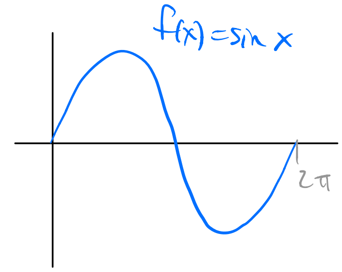
$$= \int_{x=0}^{x=4} \sqrt{u} \cdot \frac{4}{9}\, du = \frac{4}{9} \int_{x=0}^{x=4} u^{1/2}\, du = \frac{4}{9}\left[\frac{2}{3} u^{3/2}\right]_{x=0}^{x=4}$$

$$= \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\Bigg|_0^4 = \frac{8}{27} \cdot 10^{3/2} - \frac{8}{27}\, 1^{3/2} = \frac{8}{27}\left(\sqrt{1000} - 1\right) \approx 9.073$$

Remark: Often, one of these integrals $\int \sqrt{1 + (f'(x))^2}\, dx$ ends up being too complicated (e.g., no closed form sol'n, or we haven't learned the necessary method). For these, WolframAlpha is helpful.

Ex: Compute the arc length of one full cycle of $\sin x$.

$$\int_0^{2\pi} \sqrt{1 + (\cos x)^2}\, dx \approx 7.640 \quad \text{(WolframAlpha)}$$



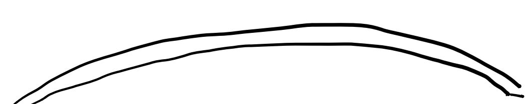
Application: Shape of an ideal arch.

What is an "ideal arch"?

Ans 1 (unsatisfactory): Satisfies 3 conditions.

1. The only load on the arch is its weight
2. The only external support is at its base
3. Gravitational forces on the arch are balanced perfectly by its reaction to the compression that these forces generate.

Ans 2 (intuition). What is not an ideal arch?

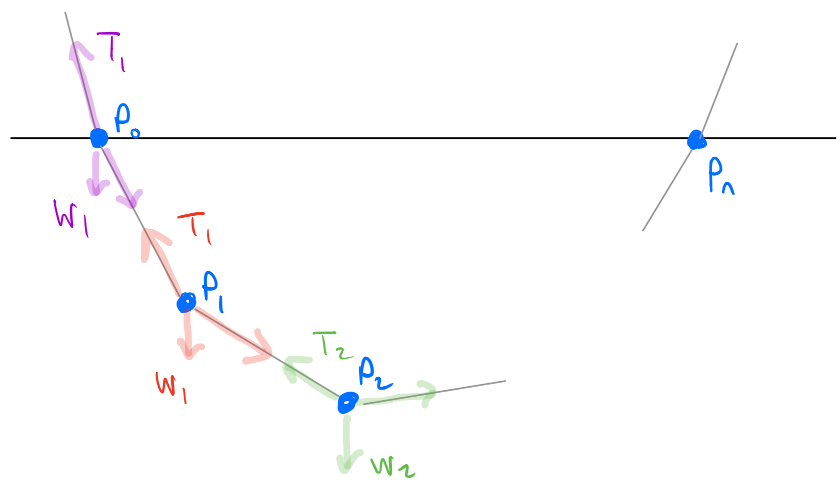


too wide

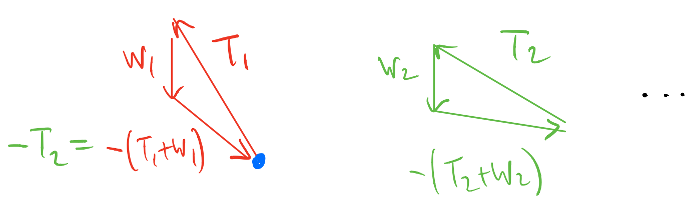
too narrow

<u>Ans 3</u>: "Hanging chain", upside-down.

Imagine a chain consisting of weights strung along (weightless) fishing line.



* Forces must balance if the string is at rest.



If the system is at rest, then we must have:

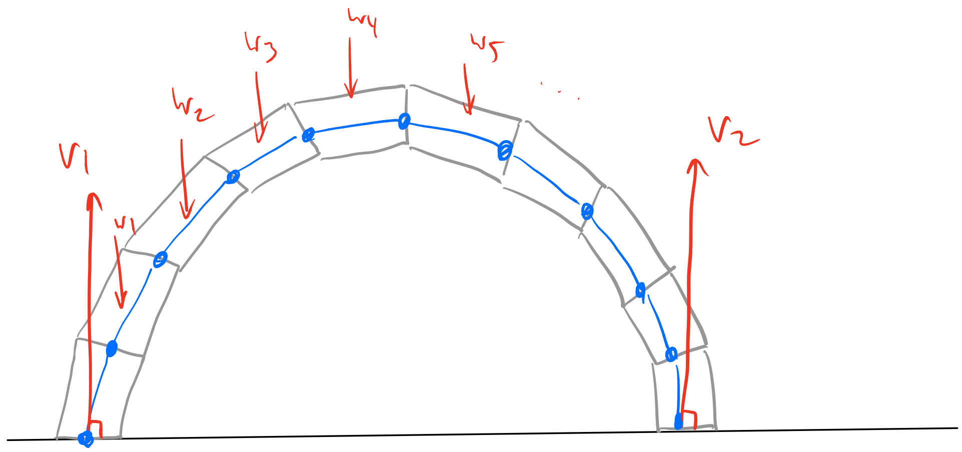
$$\begin{cases} T_0 + W_0 = T_1 \\ T_1 + W_1 = T_2 \\ \vdots \\ T_{n-1} + W_{n-1} = T_n \end{cases}$$

which is not fun to solve.

$\boxed{8}$

As usual, the "Calculus version" of this problem is easier. (Later...)

Let's turn this problem "upside-down"



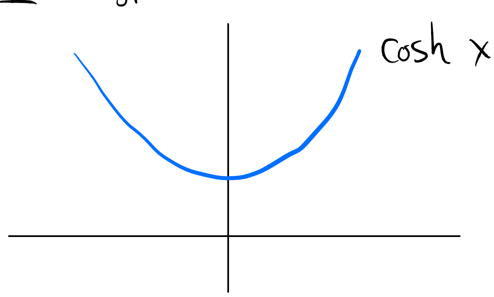
Stability $\Rightarrow$ $W_1 + W_2 + \cdots + W_n = V_1 + V_2$

<u>Safe Theorem</u> (Heyman, 1966, Cambridge Engineering): "As long as there is <u>any</u> polygonal path inside the arch such that the forces balance, as in the previous page, then the arch is stable."

It turns out that the shape of an ideal arch, or hanging chain, is a <u>hyperbolic cosine function</u>, $\boxed{\cosh(kx) = \dfrac{e^{kx}+e^{-kx}}{2}}$

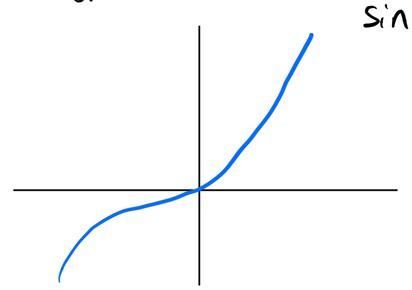
Let's take a brief detour and explore this function.

<u>Def</u>: Hyperbolic cosine



Hyperbolic sine

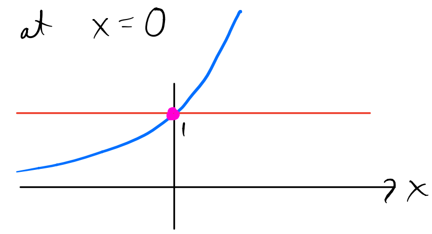
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



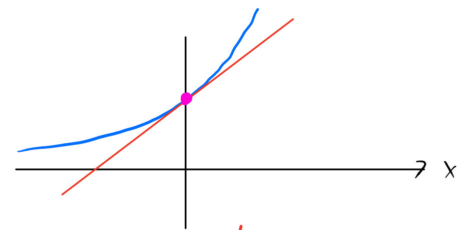
Polynomial approximations of functions (end of Calc II)

Motivating example: Let's approximate $f(x) = e^x$ at $x = 0$

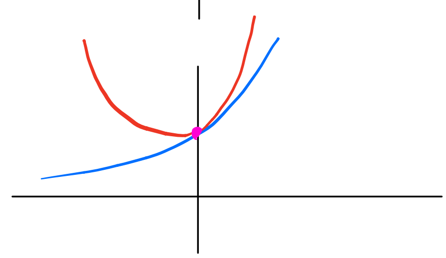
- 0^{th} order approximation: $T_0(x) = 1$



- 1^{st} order approximation: $T_1(x) = 1 + x$



- 2^{nd} order approximation: $T_2(x) = 1 + x + \frac{1}{2}x^2$



How to find this: let $T_2(x) = a_0 + a_1 x + a_2 x^2$:

$$a_0 = T_2(0) = e^0 = 1$$

$$a_1 = T_2'(0) = \frac{d}{dx}(e^x)\Big|_{x=0} = 1$$

$$2a_2 = T_2''(0) = \frac{d^2}{dx^2}(e^x)\Big|_{x=0} = 1$$

$$\vdots$$

- n^{th} order approximation: $T_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_n x^n$.

$$6a_3 = T_2'''(0) = \frac{d^3}{dx^3}(e^x) = 1$$

$$24 a_4 = T_2''''(0) = \frac{d^4}{dx^4}(e^x) = 1, \ldots$$

- Taking the limit of $T_n(x)$ as $n \to \infty$:

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots}$$

10 Week of Dec 3–7

Note: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \dots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \frac{x^8}{8!} +/- \dots$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots = \cosh x$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots = \sinh x$$

Remarks:
- $e^x = \cosh x + \sinh x$
- $\frac{d}{dx}(\sinh x) = \cosh x$, $\quad \frac{d}{dx}(\cosh x) = \sinh x$

$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} +/- \dots$

$e^{-ix} = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \frac{x^8}{8!}$

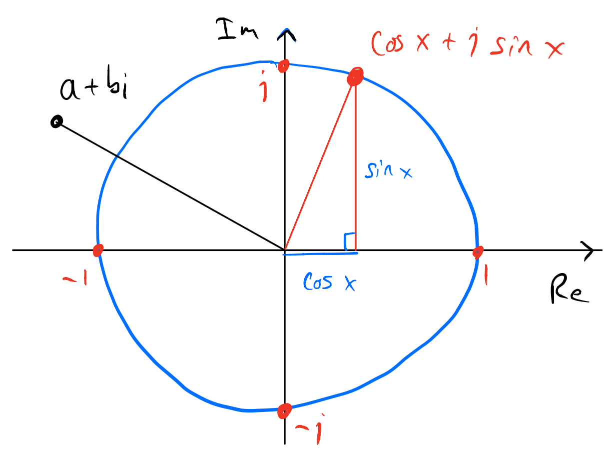
$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} +/- \dots = \cos x$$

$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +/- \dots = \sin x.$$

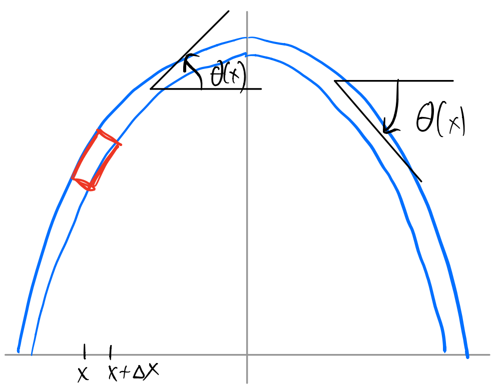
Remarks:
- $e^{ix} = \cos x + i \sin x$
- $e^{i\pi} = \cos \pi + i \sin \pi$

$= -1 + 0$

$\Rightarrow \boxed{e^{i\pi} = -1}$!!!



Analysis of an ideal arch



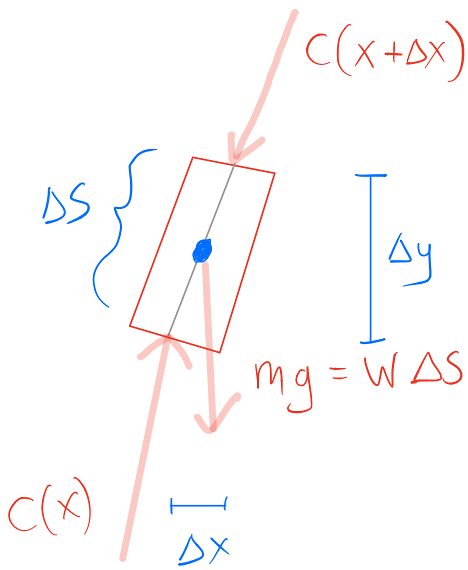
Let $C(x)$ = compression w/in arch at x.

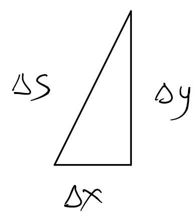
$\theta(x) = \measuredangle$ tangent makes w/ horizontal.

w = weight per unit length

Δs = length of small segment

$\Rightarrow w \cdot \Delta s$ = weight of small segment.



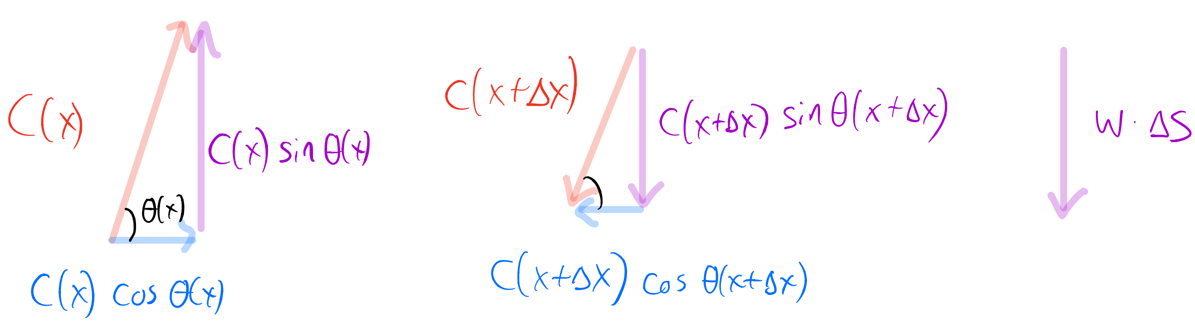


$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right] \Delta x^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

Approach: Since the arch is at rest, we will balance the horizontal & vertical forces at each segment.



12

Balance vertical forces

$$C(x)\sin\theta(x) \approx C(x+\Delta x)\sin\theta(x+\Delta x) + W\cdot\Delta S$$

$$\Rightarrow C(x+\Delta x)\sin\theta(x+\Delta x) - C(x)\sin\theta(x) \approx -W\cdot\Delta S = -w\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,dx$$

$$\Rightarrow \lim_{\Delta x\to 0}\left(\frac{C(x+\Delta x)\sin\theta(x+\Delta x) - C(x)\sin\theta(x)}{\Delta x} = -w\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x\right)$$

$$\Rightarrow \int_{-b}^{x}\left(\frac{d}{dx}C(x)\sin\theta(x) = -w\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\right)$$

$$\Rightarrow C(x)\sin\theta(x) = -w\int_{-b}^{x}\sqrt{1+(y'(x))^2}\,dx + C \qquad (\star)$$

Balance horizontal forces

$$\lim_{\Delta x\to 0}\left(\frac{C(x+\Delta x)\cos\theta(x+\Delta x) - C(x)\cos\theta(x)}{\Delta x} \approx \frac{0}{\Delta x}\right)$$

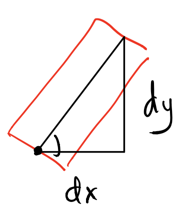
$$\int_{-b}^{x}\left(\frac{d}{dx}C(x)\cos\theta(x) = 0\right)$$

$$(\star\star) \qquad C(x)\cos\theta(x) = C_0$$

Note: Compression at top of arch is

$$= C(0)\underbrace{\cos\theta(0)}_{=1} = C_0$$

Now, consider $\tan\theta(x)$; two ways to calculate

① 

$$\tan\theta(x) = \frac{dy}{dx}$$

② $$\frac{(\star)}{(\star\star)} \qquad \frac{C(x)\sin\theta(x)}{C(x)\cos\theta(x)}$$

$$\frac{d}{dx}\left(\frac{dy}{dx} = \tan\theta(x) = \frac{\cancel{C(x)}\sin\theta(x)}{\cancel{C(x)}\cos\theta(x)} = -\frac{w}{C_0}\int_{-b}^{x}\sqrt{1+(y'(t))^2}\,dt + C\right)$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{w}{C_0}\sqrt{1+(y'(x))^2}}$$

This is a differential equation: an eq'n that implicitly defines an unknown function, $y(x)$.

All that's left: verify that $y(x) = \cosh x = \frac{e^x + e^{-x}}{2}$ satisfies this eq'n.

(In general, solving a diff eq'n is difficult; see Math 2080).

Examples of diff. eq'ns:

- $y' = y$ "what function is its own derivative?" (Ans: $y(x) = Ce^x$).
- "The rate of change of $f(t)$ is proportional to $f(t)$ itself":

  $y' = ky$. Ans: $y(x) = Ce^{kt}$