1. Let S be the following set of 7 "binary squares":

$$S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

For the following actions, draw an action diagram and find the stabilizer of each element:

(a) The action of $G = V_4 = \langle v, h \rangle$ on S, where $\phi(v)$ reflects each square vertically, and $\phi(h)$ reflects each square horizontally.

(b) The action of $G = C_4 = \langle r \rangle$ on S, where $\phi(r)$ rotates each square 90° clockwise.

2. Let $G = S_4$ act on itself by conjugation via the homomorphism

$$\phi \colon G \longrightarrow \mathrm{Perm}(S), \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

(a) How many orbits are there? Describe them as specifically as you can.

(b) Find the orbit and the stabilizer of the following elements:

i. id ii. $(1\ 2)$ iii. $(1\ 2\ 3)$ iv. $(1\ 2\ 3\ 4)$ v. $(1\ 2)\,(3\ 4)$.

3. Prove that if $H \leq S_n$ contains an odd permutation, then exactly half of its permutations are odd.

4. Suppose $H, K \leq G$ both have finite index.

(a) Show that there is some $N \trianglelefteq G$ with $N \leq H$ such that $[G : N] < \infty$.

(b) Show that $[G : H \cap K] \leq [G : H][G : K]$.

5. Let G be a finite group acting on a set S.

(a) Prove that if G has no subgroup of index 2, then any subgroup of index 3 is normal.

(b) Prove that if $[G : H] = p$, the smallest prime dividing $|G|$, then $H \trianglelefteq G$.

6. Given a left action, show that $\mathrm{Stab}_G(xs) = x\,\mathrm{Stab}_G(s)x^{-1}$ for all $x \in G$, $s \in S$.

7. Permutation groups G_1 and G_2 acting on sets S_1 and S_2 are called *permutation isomorphic* if there exists an isomorphism $\theta \colon G_1 \to G_2$ and a bijection $\phi \colon S_1 \to S_2$ such that $(\theta x)(\phi s) = \phi(xs)$ for all $x \in G_1$ and $s \in S_1$. In other words, the following diagram commutes:

$$\begin{array}{ccc} S_1 & \xrightarrow{x} & S_1 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ S_2 & \xrightarrow[\theta x]{} & S_2 \end{array}$$

Show that the following two actions of G on itself are permutation isomorphic:

(i) the action of $x \in G$ is left multiplication by x;

(ii) the action of $x \in G$ is right muliplication by x^{-1}.