1. If $|G| = 8$, then by the first Sylow theorem, G must contain a subgroup H of order 4.

   (a) If all subgroups of G of order 4 are isomorphic to V_4, then what group must G be? Completely justify your answer.

   (b) Otherwise, G has a subgroup $H \cong C_4$, and so it has a Cayley diagram like one of the following:

   Classify all groups of order 8 up to isomorphism by finding all possibilities for finishing the Cayley diagram. Fully justify the completeness of your list.

2. Show that there are no simple groups of the following order:

   (i) p^n, $(n > 1)$, (ii) pq, $(p, q$ prime) (iii) 56, (iv) 108.

3. Let P be a Sylow p-subgroup of G.

   (a) Show that P has subgroups $1 = G_0 \leq G_1 \leq \cdots \leq P_n = P$ such that $[P_i : P_{i-1}] = p$ for all $1 \leq i \leq n$.

   (b) Show that $N_G(N_G(P)) = N_G(P)$.

   (c) If $P \trianglelefteq H \trianglelefteq G$, show that $G = NH$, where $N = N_G(P)$.

4. Let G be a group of order 90 with no normal Sylow 5-subgroups.

   (a) Show that there is a nontrivial homomorphism $\varphi : G \to S_6$.

   (b) If $\varphi(G) \subseteq A_6$, show that φ is not injective.

   (c) Show that G is not simple.

5. Let G be a simple group of order 168. Show that G is isomorphic to a subgroup of A_8, the alternating group.

6. Let P be a Sylow p-subgroup of G. Show that if $x, y \in C_G(P)$ are conjugate in G, then they are conjugate in $N_G(P)$.