1. Prove that if a coproduct $(S, \{\iota_\alpha\})$ exists for a family $\{G_\alpha \mid \alpha \in A\}$ of groups, it is unique up to isomorphism and each $\iota_\alpha \colon G_\alpha \to S$ is injective.

2. Consider a category where each object X_i contains an identity element e_i (e.g., groups, abelian groups, vector spaces, etc.). Consider the Cartesian product $X_1 \times X_2$ of two objects together with the maps

$$\iota_1 \colon X_1 \to X_1 \times X_2, \qquad \iota_1(x) = (x, e_2)$$
$$\iota_2 \colon X_2 \to X_1 \times X_2, \qquad \iota_2(x) = (e_1, x).$$

   (a) Prove or disprove: this is a coproduct for $\{X_1, X_2\}$ in the category of abelian groups.

   (b) Prove or disprove: this is a coproduct for $\{X_1, X_2\}$ in the category of groups.

3. Let $\{A_\alpha \mid \alpha \in I\}$ be a collection of abelian groups, written additively.

   (a) Prove that the set of finite sums

$$\sum_{\alpha \in I} A_\alpha := \{x_{\alpha_1} + \cdots + x_{\alpha_k} \mid x_{\alpha_i} \in A_{\alpha_i}\}$$

   is a coproduct in the category of abelian groups, where ι_α are the canonical inclusion maps.

   (b) Show by example how the direct product $\prod_{\alpha \in I} A_\alpha$ fails to be a coproduct in the category of abelian groups.

4. Given groups G and H, let S be the disjoint union of their nonidentity elements. The *free product*, denoted $G * H$, is the set of all words of the form

$$G * H = \{s_1 s_2 \cdots s_k \mid s_i \in S\},$$

   where no two consecutive terms both lie in G or in H. You may assume that $G * H$ is a group where the operation is concatenation, and reduction by iterative removal of identity elements gg^{-1} and hh^{-1} that arise.

   (a) Prove that $G * H$ is a coproduct of $\{G, H\}$ in the category of groups.

   (b) Describe the group $\mathbb{Z}_2 * \mathbb{Z}_2$. What do the elements look like? Write out a group presentation (no proof needed) and construct an isomorphism to a familiar group.

5. (a) Show that if $G = G_1 G_2$ is an internal direct product, then $G/G_i \cong G_j$ for $i \neq j$.

   (b) Show that $Z(\prod_\alpha G_\alpha) = \prod_\alpha Z(G_\alpha)$.

   (c) Show that $(G_1 \times \cdots \times G_n)' = G_1' \times \cdots \times G_n'$.

   (d) Under what circumstances is $G_1 \times \cdots \times G_n$ solvable?