1. Let $\mathfrak{C}$ be a category.
   (a) If $f \in \mathrm{Hom}_{\mathfrak{C}}(A, B)$ is an isomorphism, show that there is at most one $g \in \mathrm{Hom}_{\mathfrak{C}}(B, A)$ such that $gf = 1_A$ and $fg = 1_B$.
   (b) Prove that any two initial (universal) objects in $\mathfrak{C}$ are equivalent.
   (c) Prove that any two terminal (couniversal) objects in $\mathfrak{C}$ are equivalent.

2. Let $\{A_i \mid i \in I\}$ be a family of objects in a category $\mathfrak{C}$ where products and coproducts always exist. In class, we defined a new category $\mathfrak{D}$ where the objects involved maps of the form $B \xrightarrow{f_i} A_i$, and morphisms were defined so that the product of $\{A_i \mid i \in I\}$ arose as a terminal object. Carry out an analogous construction of a category, by carefully defining its objects and morphisms, so that the coproduct of $\{A_i \mid i \in I\}$ arises as an initial object. Prove all of your claims.

3. Let A_1, A_2, A be objects in a category $\mathfrak{C}$, and $f_i \in \mathrm{Hom}(A, A_i)$ for $i = 1, 2$. Suppose that

$$\begin{array}{ccc} A & \xrightarrow{f_1} & A_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ A_2 & \xrightarrow[g_2]{} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{f_1} & A_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1'} \\ A_2 & \xrightarrow[g_2']{} & B' \end{array}$$

   are pushouts for (A, A_1, A_2, f_1, f_2). Prove that B and B' are equivalent.

4. A pair of homomorphisms $K \xrightarrow{f} G \xrightarrow{g} H$ is *exact* at G if $\mathrm{im}(f) = \ker g$. A *short exact sequence* is a sequence $1 \to K \xrightarrow{f} G \xrightarrow{g} H \to 1$ that is exact at K, G, and H.
   (a) Show that if $K \lhd G$, $f\colon K \to G$ is the inclusion map and $g\colon G \to G/K$ is the canonical quotient map, then $1 \to K \xrightarrow{f} G \xrightarrow{g} G/K \to 1$ is a short exact sequence.
   (b) Show that $1 \to K \xrightarrow{f} G \xrightarrow{g} H \to 1$ is short exact if and only if f is 1–1, g is onto, and $\mathrm{im}(f) = \ker g$. Conclude that then K is isomorphic with a normal subgroup of G and that $G/f(K) \cong H$.
   (c) Suppose $1 \to K \to G \to H \to 1$ is a short exact sequence. Show that G is solvable if and only if both K and H are solvable.
   (d) Give an example of a short exact sequence $1 \to K \to G \to H \to 1$ for which K and H are nilpotent but G is not.

5. If G is a group and $x \in G$, define the *inner automorphism* f_x by setting $f_x(y) = xyx^{-1}$ for all $y \in G$. Write $I(G)$ for the set of all inner automorphisms of G.
   (a) Show that $I(G) \leq \mathrm{Aut}(G)$.
   (b) Show that $I(G) \cong G/Z(G)$.
   (c) If $I(G)$ is abelian show that $G' \leq Z(G)$. Conclude that G is nilpotent.
   (d) Compute $\mathrm{Aut}(S_3)$.