1. Prove the lemma from class: Let N, H, K be normal subgroups of G, with $N \leq H$ and $N \leq K$. If N contains $[G, H]$ and $[G, K]$, then it also contains $[G, HK]$.

2. Let G be a nilpotent group.

   (a) Show that every subgroup of G is nilpotent.

   (b) Show that every homomorphic image of G is nilpotent.

   (c) Show that nontrivial normal subgroups of G intersect the center nontrivially.

3. Let G be a finite group in which every maximal subgroup is normal.

   (a) Prove that G is nilpotent. [*Hint*: If not, then take a non-normal Sylow subgroup $P \leq G$, and choose a maximal $M \leq G$ containing $N_G(P)$.]

   (b) Show that every maximal subgroup of G has prime index.

4. Let G be a group with $X \subseteq G$, and let A be the normal subgroup generated by X, i.e.,

$$A = \bigcap \{N \lhd G \mid X \subseteq N\} \ .$$

   Let $Y = \{gxg^{-1} \mid x \in X,\ g \in G\}$. Show that $A = \langle Y \rangle$.

5. If H and K are subgroup of G, with $H \lhd G$, $H \cap K = 1$, and $HK = G$, then G is called a semidirect product (or split extension) of H by K.

   (a) If $\sigma = (12) \in S_n$, $n \geq 2$, show that S_n is a semidirect product of A_n by $\langle \sigma \rangle$.

   (b) Show that the dihedral group D_n is a semidirect product of two of its non-trivial subgroups.

   (c) Show that the quaternion group cannot be expressed as a semidirect product of two non-trivial subgroups.

6. Suppose H and K are groups and $\phi \colon K \to \mathrm{Aut}(H)$ is a homomorphism. Let G be the Cartesian product $H \times K$ as a set, but with binary operation $(x, y)(u, v) = (x \cdot \phi(y)u, yv)$.

   (a) Show that G is a group; denote it by $G = H \rtimes_\phi K$, and call it the *external semidirect product* of H by K relative to ϕ.

   (b) Show that $H_1 = \{(x, 1) \colon x \in H\} \lhd G$, $K_1 = \{(1, y) \colon y \in K\} \leq G$, and that G is the semidirect product of H_1 by K_1.