1. Let $S = \{a\}$.
   (a) Verify that $(\mathbb{N}, +)$ is a free semigroup on S, where $\mathbb{N} = \{1, 2, 3, \dots\}$.
   (b) Show that the free group on S is an infinite cyclic group.

2. Let (F, ϕ) be a free group on a set S.
   (a) Show that $F = \langle \phi(S) \rangle$.
   (b) Prove that every nonidentity element has infinite order
   (c) If $|S| \geq 2$, show that F is not abelian.
   (d) For any $T \subseteq S$, show that there exists a normal subgroup $N \lhd F$ such that $(F/N, \pi\phi|_T)$ is a free group on T.
   (e) Show that $(\langle \phi(T) \rangle, \phi|_T)$ is a free group on T.

3. Let F be a free group and N be the subgroup generated by $\{x^n \mid x \in F,\ n \text{ a fixed integer}\}$. Show that $N \lhd F$.

4. Let F be a free object on S, and F' a free object on S' in a concrete category $\mathfrak{C}$. Prove that if $|S| = |S'|$, then F and F' are equivalent.

5. For a positive integer n, let **Nil** be the category of nilpotent groups, and let $\mathbf{Nil}_{\leq n}$ be the category of nilpotent groups of class at most n.
   (a) Show what there cannot exist a nilpotent group N generated by two elements with the property that every nilpotent group generated by two elements is a homomorphic image of N.
   (b) Prove or disprove that free objects always exist in **Nil**.
   (c) Prove or disprove that free objects always exist in $\mathbf{Nil}_{\leq n}$.