1. Let F be a free object on a set S, with $\phi\colon S \to F$, in a concrete category $\mathfrak{C}$. Construct a new category $\mathfrak{D}$, by carefully defining $\mathrm{Ob}(\mathfrak{D})$ and $\mathrm{Hom}(\mathfrak{D})$, so that $\phi\colon S \to F$ arises as either an initial (universal) object or as a terminal (couniversal) object. Justify your claims.

2. Prove what group is described by each presentation.
   (a) $G = \langle a, b \mid a^2 = b^2 = (ab)^2 = 1\rangle$.
   (b) $G = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ ab = ba^3\rangle$.
   (c) $G = \langle a, b \mid a^2 = 1,\ b^3 = 1,\ ab = ba\rangle$.
   (d) $G = \langle a, b \mid a^4 = b^3 = 1,\ ab = ba^3\rangle$.
   (e) $G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^3 = 1\rangle$.

3. Consider the Cartesian product $H = \mathbb{Z}_2 \times \mathbb{Z}_n$ (as a *set*) with binary operation defined as

$$(\bar{\imath}, \bar{\jmath}) \cdot (\bar{k}, \bar{\ell}) = (\bar{\imath} + \bar{k}, (-1)^k \bar{\jmath} + \bar{\ell}).$$

   (a) Show that H is a group under this operation, and determine its order.
   (b) Let $G = \langle a, b \mid a^n = 1,\ b^2 = 1,\ abab^{-1} = 1\rangle$. Show that $|G| \leq 2n$.
   (c) Show that $H \cong G$.

4. Define the *generalized quaternion group* Q_n by the presentation

$$Q_n = \langle a, b \mid a^{2n} = 1,\ b^2 = a^n,\ ab = ba^{-1}\rangle, \qquad \text{for } n \geq 1\ .$$

   (a) Show that $|Q_n| = 4n$.
   (b) Show that Q_1 is cyclic and Q_2 is the quaternion group.
   (c) Show that Q_3 is not isomorphic with either D_6 or A_4.
   (d) Find the nilpotency class of Q_{2^n}.

5. Let S be a set. The group with presentation $\langle S \mid R\rangle$ where $R = \{[s, t] \mid s, t \in S\}$ is called the *free abelian* group on S; denote it by A_S and let $\phi\colon S \to A_S$ be the inclusion.
   (a) Prove that A_S has the following universal property: for any abelian group G and function $\theta\colon S \to G$, there is a unique homomorphism $f\colon A_S \to G$ such that $f\phi = \theta$.
   (b) Deduce that if A is a free abelian group on a set of cardinality n, then

$$A \cong \mathbb{Z} \times \cdots \times \mathbb{Z} \qquad (n \text{ factors}).$$