1. Let I and J be ideals of a ring R.
   (a) Prove that $I+J$, $I \cap J$, and IJ are ideals of R.
   (b) If R is commutative, then the set $(I : J) := \{r \in R \mid rJ \subset I\}$ is called the *ideal quotient* or *colon ideal* of I and J. Show that $(I : J)$ is an ideal of R.
   (c) Consider the ideals $I = 4\mathbb{Z}$ and $J = 6\mathbb{Z}$ of the ring $R = \mathbb{Z}$. Compute $I+J$, $I \cap J$, IJ, $(I : J)$, and $(J : I)$.
   (d) Repeat Part (c) for the ideals $I = m\mathbb{Z}$ and $J = n\mathbb{Z}$ of $R = \mathbb{Z}$.

2. Let $f : R \to S$ be a ring homomorphism between commutative rings.
   (a) If f is surjective and I is an ideal of R, show that $f(I)$ is an ideal of S.
   (b) Show that Part (a) is not true in general when f is not surjective.
   (c) Show that if f is surjective and R is a field, then S is a field as well.

3. If F is a field, show that the ring $M_n(F)$ of $n \times n$ matrices over F is simple.

4. Let p be a fixed prime number, and consider the ring
$$R = \left\{ \tfrac{a}{b} \mid a, b \in \mathbb{Z},\ (a,b) = 1,\ p \nmid b \right\}$$
   with the usual operations of addition and multiplication of rational numbers.
   (a) Determine the group of units of R.
   (b) Prove that the principal ideal $(p) = pR$ is a maximal ideal of R, and in fact, the only maximal ideal of R.

5. An element a of a ring R is called *nilpotent* if $a^n = 0$ for some positive integer n.
   (a) Show that the set of nilpotent elements in a commutative ring R forms an ideal.
   (b) If $u \in R$ is a unit and $a \in R$ nilpotent, show that $u + a$ is a unit.
   (c) Find $U(R)$ if $R = \mathbb{Z}_{p^k}[x]$, where p is prime and $k \geq 1$. Prove all of your claims.