Unless otherwise specified, assume that R is a commutative ring with 1.

1. Let P be a non-zero prime ideal.
   (a) Show that if $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.
   (b) Show that if R is a PID, then P is maximal.

2. For a semigroup $S \subseteq R$ containing no zero divisors, let $X = R \times S$ and define a relation where $(a, b) \sim (c, d)$ if $ad = bc$.
   (a) Show that $\sim$ is an equivalence relation on X.
   (b) Denote the equivalence class of (a, b) by a/b and the set of equivalence classes by R_S (called the *localization* of R at S). Show that R_S is a commutative ring with 1.
   (c) If $a \in S$, show that $\{ra/a \mid r \in R\}$ is a subring of R_S and that $r \mapsto ra/a$ is a monomorphism, so that R can be identified with a subring with R_S.
   (d) Show that every $s \in S$ is a unit in R_S.
   (e) Give a "universal" definition for R_S and show that it is unique up to isomorphism.

3. A *local ring* is a commutative ring with identity which has a unique maximal ideal.
   (a) Show that if x is contained in every maximal ideal of R, then $1 + x$ is a unit.
   (b) Prove that a commutative ring is local if and only if its set of non-units is an ideal.
   (c) If P is a prime ideal of an integral domain R, show that $S = R \setminus P$ is a multiplicative semigroup and R_S is local.

4. For each of the rings R and multiplicative semigroups S, construct the localization R_S and describe the elements, units, and all maximal ideals.
   (a) $R = \mathbb{Z}$ and $S = R \setminus (5)$.
   (b) $R = \mathbb{Z}$ and $S = (5) \setminus \{0\}$.
   (c) $R = \mathbb{Z}$ and $S = \{5^k \mid k \in \mathbb{N}\}$.
   (d) $R = \mathbb{Z}[x]$ and $P = R \setminus (x)$.

5. Find the field of fractions of the following rings. Prove your claims.
   (a) R is any field.
   (b) $R = 2\mathbb{Z}$.
   (c) $R = \mathbb{Z}[i] := \{a + bi \mid a, b \in \mathbb{Z}\}$.

6. Use Zorn's lemma to show that the ring $\mathbb{R}$ contains a subring A containing 1 that is maximal with respect to the property that $1/2 \notin A$, and then find its field of fractions.