Unless otherwise specified, assume that R is a commutative ring with 1 and F is a field.

1. For each of the following ideals I of $\mathbb{Z}[x]$, determine whether they are principal, and then find $I \cap \mathbb{Z}$ and $\mathbb{Z}[x]/I$, up to isomorphism.
   (a) $I = (x^2+1,\ x^2+x+3)$,
   (b) $I = (2,\ x^2+x+1)$.

2. Fix $a \in \mathbb{Z}$ and let $\phi_a : \mathbb{Z}[x] \to \mathbb{Z}$ be the evaluation map, $\phi_a(f(x)) = f(a)$.
   (a) If $I = (3, x^2+1)$ in $\mathbb{Z}[x]$, show that $\phi_a(I) = \mathbb{Z}$.
   (b) Show by example how Part (a) can fail if 3 is replaced with a different odd prime p.

3. (a) Show that an ideal $I = (f)$ of $F[x]$ is maximal if and only if f is irreducible.
   (b) Show by example how Part (a) can fail in $R[x]$, where R is not a field.
   (c) Consider the ideal $I = (x_1 - a_1, \dots, x_n - a_n)$ in $R[x_1, \dots, x_n]$, for distinct $a_1, \dots, a_n \in R$. Formulate and prove necessary and sufficient conditions on R that will ensure that I is a maximal ideal.

4. For $f(x) \in R[x]$, substituting $a \in R$ for x determines a *polynomial function* $f\colon R \to R$, where $a \mapsto f(a)$.
   (a) If R is an infinite integral domain, show that the mapping $f(x) \mapsto f$ assiging to each polynomial in $R[x]$ to its corresponding polynomial function is 1–1.
   (b) If R is finite, show that there must exist distinct polynomials $f(x)$ and $g(x)$ in $R[x]$ with identical associated functions f and g, i.e., $f(a) = g(a)$ for all $a \in R$.

5. Let $\mathbb{F}_q$ be the finite field of order $q = p^n$.
   (a) Show that every function $f\colon \mathbb{F}_q \to \mathbb{F}_q$ can be written uniquely as a polynomial in the quotient $\mathbb{F}_q[x]/I$, where $I = (x^q - x)$. How many such functions are there?
   (b) Generalize Part (a) to multivariate functions $f\colon \mathbb{F}_q^n \to \mathbb{F}_q$.
   (c) A *Boolean network* is an n-tuple $(f_1, \dots, f_n)$ of functions $f_i\colon \mathbb{F}_2^n \to \mathbb{F}_2$. How many 6-node Boolean networks are there? Express your answer in scientific notation.