Unless otherwise specified, assume that R is a commutative ring with 1.

1. For an ideal I of $R[x]$ and $m \in \mathbb{N}$, let $I(m)$ denote the set of leading coefficients of degree-m polynomomials in I, together with 0.
   (a) Show that $I(m)$ is an ideal in R.
   (b) Show that $I(m) \subseteq I(m+1)$ for all m.
   (c) If J is an ideal with $I \subseteq J$, show that $I(m) \subseteq J(m)$ for all m.

2. Consider the following rings R_i, for $i = 1, \ldots, 6$, which are additionally $\mathbb{C}$-vector spaces:

$$R_1 = \mathbb{C}[x]/(x^3 - 1)$$
$$R_2 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$$
$$R_3 = \text{the ring of upper triangular } 2 \times 2 \text{ matrices over } \mathbb{C}$$
$$R_4 = \mathbb{C}[x]/(x-1) \times \mathbb{C}[x]/(x+i) \times \mathbb{C}[x]/(x-i)$$
$$R_5 = \mathbb{C}[x]/(x^2+1) \times \mathbb{C}[x]/(x-1)$$
$$R_6 = \mathbb{C}[x]/(x+1)^2 \times \mathbb{C}[x]/(x-1).$$

   (a) Compute the dimension of each R_i as a $\mathbb{C}$-vector space by giving an explicit basis.
   (b) Partition the rings $R_1, \ldots, R_6$ into isomorphism classes.

3. (a) Solve the congruences

$$x \equiv 1 \pmod{8}, \qquad x \equiv 3 \pmod{7}, \qquad x \equiv 9 \pmod{11}$$

   simultaneously for x in the ring $\mathbb{Z}$ of integers.
   (b) Solve the congruences

$$x \equiv i \pmod{i+1}, \qquad x \equiv 1 \pmod{2-i}, \qquad x \equiv 1+i \pmod{3+4i}$$

   simultaneously for x in the ring R_{-1} of Gaussian integers.
   (c) Solve the congruences

$$f(x) \equiv 1 \pmod{x-1}, \qquad f(x) \equiv x \pmod{x^2+1}, \qquad f(x) \equiv x^3 \pmod{x+1}$$

   simultaneously for $f(x)$ in $F[x]$, where F is a field in which $1 + 1 \neq 0$.

4. An element $e \in R$ is called an *idempotent* if $e^2 = e$, and two nonzero idempotents e_1, e_2 are called an *orthogonal pair* if $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.
   (a) Show that the following are equivalent:
      (i) R contains an idempotent different from 0 and 1.
      (ii) R contains an orthogonal pair of idempotents.
      (iii) $R \cong R_1 \times R_2$ for some rings R_1 and R_2.
   (b) Give an example of a non-orthogonal pair of distinct idemponents.
   (c) Find all idempotents in the ring $\mathbb{Z}/20\mathbb{Z}$.