5. Products, coproducts, and categories

We'll motivate the difference between products & coproducts with vector spaces.

Question: What is $\lim_{n\to\infty} \mathbb{R}^n$? ("infinite dimensional space")

There are several things this could mean:

Product: $\mathbb{R}^\infty := \prod_{i=1}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$ contains $(1,1,1,\dots)$

Sum: $\mathbb{E}^\infty := \mathbb{R}\vec{e}_1 \oplus \mathbb{R}\vec{e}_2 \oplus \mathbb{R}\vec{e}_3 \oplus \cdots$ contains $5e_1 - 3e_2 + \frac{4}{3}e_7$

"finite sums $v_{i_1} + \cdots + v_{i_k}$" $\quad (5, -3, 0, 0, 0, 0, \frac{4}{3}, 0, 0, \dots)$

Note: There is a canonical embedding $\mathbb{E}^\infty \hookrightarrow \mathbb{R}^\infty$.

Def: If $\{X_\alpha : \alpha \in A\}$ is a non-empty family of [objects], then a **product** of the X_α's is a pair $(P, \{p_\alpha\})$ of an [object] P and [morphisms] $p_\alpha : P \to X_\alpha$ satisfying the following universal property:

Given any [object] H and [morphisms] $f_\alpha : H \longrightarrow X_\alpha$, $\alpha \in A$, $\exists!\ f : H \longrightarrow P$ s.t. $p_\alpha f = f_\alpha \quad \forall \alpha \in A$.

A **coproduct** of the X_α's is a pair $(S, \{\iota_\alpha\})$ of an [object] S and [morphisms] $\iota_\alpha : X_\alpha \longrightarrow S$ satisfying the following universal property:

Given any [object] H and [morphisms] $f_\alpha : X_\alpha \longrightarrow H$, $\alpha \in A$, $\exists!\ f : S \longrightarrow H$ s.t. $f\iota_\alpha = f_\alpha \quad \forall \alpha \in A$.

$$\begin{tikzcd}
H \arrow[rr, "f_\alpha"] \arrow[dr, dashed, "F"'] \arrow[rr, phantom, shift right=3ex, "\text{product}"] & & X_\alpha \\
& P \arrow[ur, "p_\alpha"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_\alpha \arrow[rr, "f_\alpha"] \arrow[dr, "\iota_\alpha"'] \arrow[rr, phantom, shift right=3ex, "\text{coproduct}"] & & H \\
& S \arrow[ur, dashed, "f"'] &
\end{tikzcd}$$

<u>In words</u>:

- The <u>product</u> of the X_α's is "the smallest" object that projects into each factor X_α.
- The <u>coproduct</u> (or <u>sum</u>) of the X_α's is "the smallest" object into which each factor X_α maps.

☆ Usually, $p_\alpha : P \twoheadrightarrow X_\alpha$ and $\iota_\alpha : X_\alpha \hookrightarrow S$.

Back to vector spaces for motivation. [Let $X_j = \mathbb{R}\vec{e}_j \cong \mathbb{R}$].

$$\begin{tikzcd}
& H \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[d, dashed, "F"] & \\
X_1 & X_1 \times X_2 \times \cdots \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & X_2
\end{tikzcd}
\qquad
\begin{tikzcd}
& H & \\
X_1 \arrow[ur, "f_1"] \arrow[r, hook, "\iota_1"'] & X_1 \oplus X_2 \oplus \cdots \arrow[u, dashed, "f"] & X_2 \arrow[ul, "f_2"'] \arrow[l, hook', "\iota_2"]
\end{tikzcd}$$

What does <u>not</u> work:

$$\begin{tikzcd}
& X_1 \times X_2 \times \cdots \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[d, dashed, "F"] & \\
X_1 & \underbrace{X_1 \oplus X_2 \oplus \cdots}_{\text{"too small"}} \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & X_2
\end{tikzcd}
\qquad
\begin{tikzcd}
& X_1 \oplus X_2 \oplus \cdots & \\
X_1 \arrow[ur, "f_1"] \arrow[r, hook, "\iota_1"'] & \underbrace{X_1 \times X_2 \times \cdots}_{\text{"too big"}} \arrow[u, dashed, "f"] & X_2 \arrow[ul, "f_2"'] \arrow[l, hook', "\iota_2"]
\end{tikzcd}$$

☆ How to define $(1,1,1,\ldots) \overset{F}{\longmapsto}$???

Note that this construction works regardless of the number of factors: finite, countable, uncountable, etc.

We'll focus on the case where $X_\alpha = G_\alpha$ is a <u>group</u>, and all maps are homomorphisms.

<u>Prop 5.1</u>: If a product $(P, \{p_\alpha\})$ exists for $\{G_\alpha : \alpha \in A\}$, it is unique up to isomorphism, and each $p_\alpha : P \to G_\alpha$ is onto.

<u>Pf</u>: Let $(P', \{p_\alpha'\})$ be another product. We have, for each $\alpha \in A$:

$$\begin{tikzcd} P' \arrow[r, "p_\alpha'"] \arrow[dr, dashed, "f"'] & G_\alpha \\ & P \arrow[u, "p_\alpha"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \arrow[r, "p_\alpha"] \arrow[dr, dashed, "g"'] & G_\alpha \\ & P' \arrow[u, "p_\alpha'"'] \end{tikzcd}$$

If we "stack" the diagrams, we get for each $\alpha \in A$:

$$\begin{tikzcd} P \arrow[rr, "p_\alpha"] \arrow[dr, dashed, "g"'] & & G_\alpha \\ & P' \arrow[ur, "p_\alpha'"] \arrow[dr, dashed, "f"'] & \\ & & P \arrow[uu, "p_\alpha"'] \end{tikzcd} \quad \text{but we also know:} \quad \begin{tikzcd} P \arrow[r, "p_\alpha"] \arrow[dr, dashed, "1_P"'] & G_\alpha \\ & P \arrow[u, "p_\alpha"'] \end{tikzcd}$$

$p_\alpha = p_\alpha' = p_\alpha f g$ $\qquad\qquad$ $p_\alpha = p_\alpha 1_P$

By uniqueness, $fg = 1_P$. Similarly, $gf = 1_{P'}$, so f and g are inverse isomorphisms. ✓

Next, we'll show each p_α is onto.

For each G_α, define $f_\beta : G_\alpha \to G_\beta$, to be

$\boxed{4}$

$$f_\beta(x) = \begin{cases} 1 \in G_\beta & x \in G_\beta \neq G_\alpha \quad \text{trivial map} \\ x \in G_\beta & x \in G_\beta = G_\alpha \quad \text{identity map} \end{cases}$$

Fix $H = G_\alpha$. For all $\beta \in A$,

$$\begin{tikzcd} H = G_\alpha \arrow[rr, "f_\beta"] \arrow[dr, dashed, "f"'] \arrow[r, phantom, "\beta \in A"] & & G_\beta \\ & P \arrow[ur, "p_\beta"'] & \end{tikzcd} \quad \xrightarrow[\text{of } \beta = \alpha]{\text{special case}} \quad \begin{tikzcd} G_\alpha \arrow[rr, "f_\alpha = 1_\alpha"] \arrow[dr, dashed, "f"'] \arrow[r, phantom, "\beta = \alpha"] & & G_\alpha \\ & P \arrow[ur, "p_\alpha"'] & \end{tikzcd}$$

Since $1_\alpha = p_\alpha f$ and 1_α is onto, so is p_α. $\square$

Thm 5.2: If $\{G_\alpha : \alpha \in A\}$ is a nonempty family of groups, then the product of $\{G_\alpha : \alpha \in A\}$ exists.

Pf: We'll show that the Cartesian product is the product.

Let $P = \prod_{\alpha \in A} G_\alpha$.

Write elements as $(x_\alpha)_{\alpha \in A}$, where $(x_\alpha)(y_\alpha) := (x_\alpha y_\alpha)$.

Define $p_\alpha : P \longrightarrow G_\alpha$ as the "projection map"

$(x_\beta) \longmapsto x_\alpha$.

Suppose $f_\alpha : H \longrightarrow G_\alpha$ is a homomorphism, for all $\alpha \in A$.

Define $f : H \longrightarrow P$, $f(h) = (f_\alpha(h))_{\alpha \in A}$

Check: $p_\alpha f = f_\alpha$

$$\begin{tikzcd} H \arrow[rr, "f_\alpha"] \arrow[dr, dashed, "f"'] & & G_\alpha \\ & P \arrow[ur, "p_\alpha"'] & \end{tikzcd} \qquad \begin{tikzcd} h \arrow[rr, mapsto, "f_\alpha"] \arrow[dr, dashed, mapsto, "f"'] & & f_\alpha(h) \\ & (f_\alpha(h))_{\alpha \in A} \arrow[ur, "p_\alpha"'] & \end{tikzcd}$$

Uniqueness: Consider
$$\begin{tikzcd} H \arrow[r, "f_\alpha"] \arrow[dr, dashed, "f"'] & G_\alpha \\ & P \arrow[u, "p_\alpha"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H \arrow[r, "f_\alpha"] \arrow[dr, dashed, "g"'] & G_\alpha \\ & P \arrow[u, "p_\alpha"'] \end{tikzcd}$$

i.e., $p_\alpha f = p_\alpha g$.

Then $(f(x))_\alpha = p_\alpha f(x) = p_\alpha g(x) = (g(x))_\alpha$

$\Rightarrow f(x) = g(x) \quad \forall x \in H.$ ✓ $\square$

If $A = \{1, 2, \ldots, n\}$ or $\{1, 2, 3, \ldots\}$, we write $G_1 \times G_2 \times \cdots \times G_n$ or $G_1 \times G_2 \times G_3 \times \cdots$. This is the direct product, denoted $\prod_{\alpha \in A} G_\alpha$.

The homomorphism $p_\alpha : \prod_{\alpha \in A} G_\alpha \longrightarrow G_\alpha$ is the projection of the product onto the direct factor G_α.

Thm 5.3: Suppose $G_1, G_2 \leq G$ satisfy:

(i) $G_1, G_2 \trianglelefteq G$

(ii) $G_1 \cap G_2 = 1$

(iii) $G_1 G_2 = G$.

Then $G \cong G_1 \times G_2$.

Pf: (Sketch): Since $G_1 \cap G_2 = 1$, each $x = x_1 x_2$ uniquely, with $x_i \in G_i$.

Also, $[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2 \in G_1 \cap G_2 = 1 \Longrightarrow x_1 x_2 = x_2 x_1$.

Define the projection maps $p_i(x) = x_i$, $i = 1, 2$ (check homom!)

Now, for any homomorphisms $f_i : H \longrightarrow G_i \quad i = 1, 2$

define $f : H \longrightarrow G$, $\quad f(h) = f_1(h) f_2(h)$.

Check that $p_i f = f_i$ ✓

Uniqueness: Suppose $g: H \to G$ also satisfies $p_i g = f_i$ (for each i), then $p_i f = p_i g$. Then, for any $x \in H$, we have

$$f(x) = p_1(f(x))\, p_2(f(x)) = p_1(g(x))\, p_2(g(x)) = g(x).$$

Since products exist and are unique, and G is a product, it follows that $G \cong G_1 \times G_2$. $\square$

Remark: Thm 5.3 naturally extends to $n > 2$ factors. Condition (ii) $(G_1 \cap G_2 = 1)$ becomes: $G_i \cap \left\langle \bigcup_{j \neq i} G_j \right\rangle = 1$ for each i.

When the assumptions of Thm 5.3 hold, we say that G is the internal direct product of its subgroups G_1, G_2 (or $G_1, \dots, G_n$).

Some basic category theory

Def: A category $\mathcal{C}$ consists of:

(I) A class of objects $Ob(\mathcal{C})$

(II) A class of morphisms $Hom(\mathcal{C})$ between objects, with

- (i) Identity morphism $1_A : A \to A$ for all $A \in Ob(\mathcal{C})$
- (ii) Composition: $f \in Hom_{\mathcal{C}}(A,B)$, $g \in Hom_{\mathcal{C}}(B,C) \Rightarrow g \circ f \in Hom_{\mathcal{C}}(A,C)$.
- (iii) Associative: $h \circ (g \circ f) = (h \circ g) \circ f$.

Think of a category as a directed multi-graph:

vertices $\longleftrightarrow$ objects edges $\longleftrightarrow$ morphisms.

Def: A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is a:

- **monomorphism** if $fg_1 = fg_2 \Rightarrow g_1 = g_2$
- **epimorphism** if $g_1 f = g_2 f \Rightarrow g_1 = g_2$
- **isomorphism** if $\exists\, g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$ with $fg = 1_B$ and $gf = 1_A$

  In this case, we say that A and B are **equivalent**.

Remark: Given a collection of objects $\{X_\alpha : \alpha \in A\}$, we can define their **product** and **coproduct**; see the beginning of these notes.

Examples: (See HW for details)

Category	Objects	Morphisms	Product	Coproduct
Set	Sets	Functions	Cartesian prod.	Disjoint union
Top	Top. spaces	Contin. maps	Cartesian prod.	Disjoint union
Grp	Groups	Homomorphisms	Direct product	Free product
Ab	Abelian groups	Homomorphisms	Direct product	Direct sum
Vect_K	K-vector spaces	Linear maps	Direct product	Direct sum

Remark: Any poset defines a category.

Products and coproducts are defined via <u>universal mapping properties</u> i.e, by the existence of certain uniquely determined morphisms.

This can be generalized.

Def:
- An object $I \in Ob(\mathcal{C})$ is universal (or initial) if for each $C_i \in Ob(\mathcal{C})$, $\exists!\ p_i \in Hom_{\mathcal{C}}(I, C_i)$.
- An object $T \in Ob(\mathcal{C})$ is couniversal (or terminal) if for each $C_i \in Ob(\mathcal{C})$, $\exists!\ L_i \in Hom_{\mathcal{C}}(C_i, T)$.
- An initial and terminal object is a zero object.

Examples	Category	Universal (initial)	Couniversal (terminal)
	Set	$\varnothing$	$\{x\}$ (any x)
	Top	$\varnothing$	$\{x\}$
	Grp	1	1

Thm 5.4: Any two universal objects are equivalent.

Pf: (sketch) Let I and J be universal.

$$\begin{tikzcd}
I \arrow[rr, "g\circ f = 1_I"] \arrow[dr, "\exists! f"'] & & I \\
& J \arrow[ur, "\exists! g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
J \arrow[rr, "f\circ g = 1_J"] \arrow[dr, "\exists! g"'] & & J \\
& I \arrow[ur, "\exists! f"'] &
\end{tikzcd}$$

Thus $fg = 1_J$ and $gf = 1_I \Longrightarrow I$ & J are equivalent. $\square$

Similarly, we can show that any two couniversal objects are equivalent. (HW)

☆ For suitably chosen categories (co)products are the (co)universal objects.

<u>Example</u>: Let $\{A_i \mid i \in I\}$ be a family of objects in $\mathcal{C}$.

Define a new category $\mathcal{D}$ as follows:

- <u>Objects</u>: Pairs $(B, \{f_i \mid i \in I\})$ where $f_i \in \mathrm{Hom}_{\mathcal{C}}(B, A_i)$.
- <u>Morphisms</u>: Elements $h \in \mathrm{Hom}_{\mathcal{C}}(B, D)$ s.t. $g_i \circ h = f_i \quad \forall i \in I$

$$\begin{tikzcd} B \arrow[rr, "f_i"] \arrow[dr, dashed, "h"'] & & A_i \\ & D \arrow[ur, "g_i"'] & \end{tikzcd}$$

<u>Check</u>: In this category, the couniversal (terminal) object is

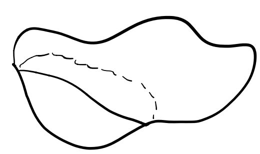
$\left(\prod_{i \in I} A_i \,,\ \{p_i \mid i \in I\}\right)$. [If $\prod_{i\in I} A_i$ exists in $\mathcal{C}$.]

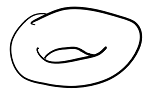
<u>Cor</u>: Products and coproducts are unique up to equivalence (when they exist).

<u>Short aside: topology</u>

Topology can be thought of as "analysis & geometry without the metric."

Two spaces are <u>homeomorphic</u> if one can be continuously deformed to the other, and vice-versa.

For example:  $\simeq$  are spheres

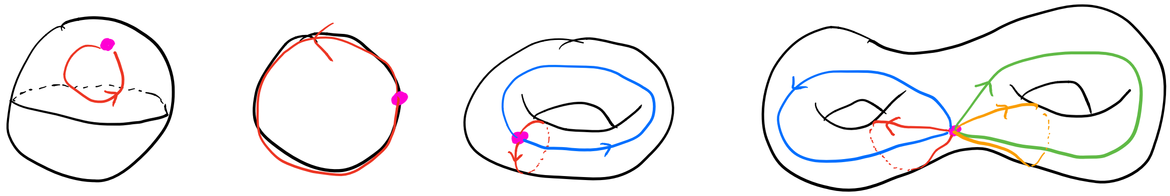
 $\simeq$  are tori (plural of "torus").

Think about fundamental differences b/w the sphere and torus

e.g., 4-color theorem, vs. 7-color theorem.

any 2 great circles intersect twice, vs. some geodesics won't intersect.

$\boxed{10}$ Every topological space X has an associated <u>fundamental group</u>, $\pi_1(X)$, consisting of closed loops up to continuous deformation.



$\pi_1(S^2) \cong 1$ $\quad$ $\pi_1(S^1) \cong \mathbb{Z}$ $\quad$ $\pi_1(T^2) = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z} \times \mathbb{Z}$ $\quad$ $\pi_1(T^2) = \langle a,b,c,d \mid ab = ba,\ cd = dc \rangle$

<u>Key idea</u>: There is a "<u>functor</u>" between categories $\pi_1 : \mathrm{Top} \longrightarrow \mathrm{Grp}$ that preserves the structure of morphisms.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\pi_1"] \\
\pi_1(X) \arrow[r, "f_*"] & \pi_1(Y)
\end{tikzcd}$$

f is a continuous map

f_* is the "induced homomorphism".

This also preserves products and coproducts

<u>Def</u>: Let A, A_1, A_2, be objects in a category $\mathcal{C}$ and let $f_i \in \mathrm{Hom}_{\mathcal{C}}(A, A_i)$ for $i = 1, 2$. A <u>pushout</u> (or fiber coproduct) for (A, A_1, A_2, f_1, f_2) is a commutative diagram with the following property:

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \arrow[d, "g_1"] \arrow[ddr, bend left, "h_1"] & \\
A_2 \arrow[r, "g_2"'] \arrow[drr, bend right, "h_2"'] & B \arrow[dr, dashed, "h"] & \\
 & & C
\end{tikzcd}$$

☆ for any object $C \in \mathrm{Ob}(\mathcal{C})$ and morphisms $h_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, C)$ s.t. if $h_1 f_1 = h_2 f_2$, $\exists!\ h \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ s.t. $h g_i = h_i$ for $i = 1, 2$.

Prop: Suppose

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \arrow[d, "g_1'"] \\ A_2 \arrow[r, "g_2'"'] & B' \end{tikzcd}$$

is another pushout for (A, A_1, A_2, f_1, f_2).
Then $B' \cong B$.

Pf: HW.

Examples

1. $\mathcal{C} = \mathrm{Set}$. let $A = A_1 \cap A_2$
   $f_i : A \hookrightarrow A_i$.

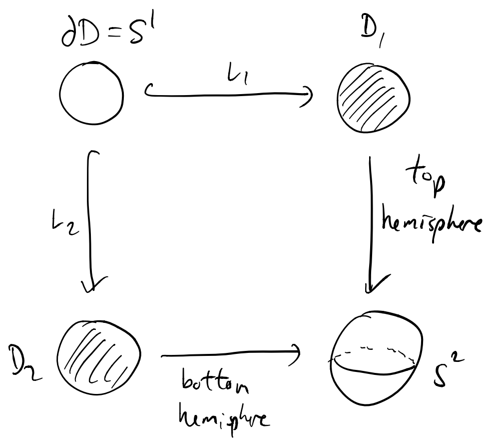
Pushout:

$$\begin{tikzcd} A_1 \cap A_2 \arrow[r, hook, "f_1"] \arrow[d, hook, "f_2"'] & A_1 \arrow[d, hook, "g_1"] \\ A_2 \arrow[r, hook, "g_2"'] & A_1 \cup A_2 \end{tikzcd}$$

Think: $A_1 \sqcup A_2$ with $g_1(A_1)$ & $g(A_2)$ identified.

2. $\mathcal{C} = \mathrm{Top}$ (or Set)

Consider 2 disjoint disks, D_1 & D_2.
They have boundary circle $\partial D_i = S^1$.
The pushout of $(S^1, D_1, D_2, \iota_1, \iota_2)$,
where $\iota_i : S^1 \hookrightarrow D_i$ are inclusion maps,
is the 2-sphere.



Think: "glue 2 disks along their boundary circle."

Question: what do we get if $\iota_2 : S^1 \to D_2$ is the map that "wraps S^1 around ∂D_2 twice"?

3 $\mathcal{C} = \mathrm{Grp}$. If $f_{1,2}$ are injective, then the pushout is the free product with amalgamation, and is the quotient $G_1 *_H G_2 = (G_1 * G_2)/N$, where $N = \langle f_1(h) f_2(h)^{-1} : h \in H \rangle$.

$$\begin{tikzcd} H \arrow[r, hook, "f_1"] \arrow[d, hook, "f_2"'] & G_1 \arrow[d, "g_1"] \\ G_2 \arrow[r, "g_2"'] & B \end{tikzcd}$$

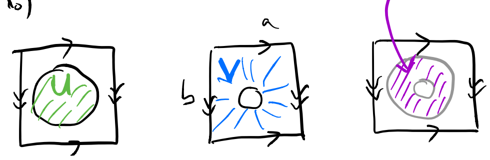
Note: If $H = 1$, this is just the free product, $G_1 * G_2$, the coproduct.

Application: Seifert-Van Kampen theorem (algebraic topology).

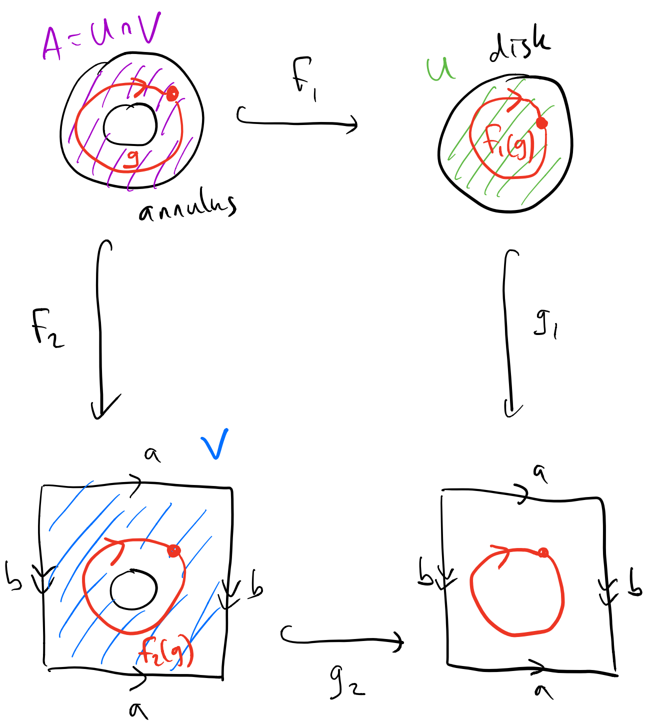
Sketch of main ideas: Let $X = U \cup V$ and $A = U \cap V$ be path-connected top. spaces. The fundamental group of X is

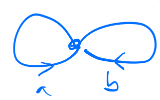
$$\pi_1(X, x_0) \cong \pi_1(U, x_0) \underset{\pi_1(A, x_0)}{*} \pi_1(V, x_0).$$

Example: $X = T^2$ (torus)  $= U \cup V$.



$\pi_1(U) \cong 1$ $\qquad$ $\pi_1(U \cap V) \cong \mathbb{Z}$



V deformation-retracts to a wedge of two circles $S^1 \vee S^1$: ,

with fundamental group

$\pi_1(V, x_0) = \langle a, b \mid \ \rangle \cong \mathbb{Z} * \mathbb{Z}$

$*$ Siefert-Van Kampen says this pushout of top. spaces carries over to a pushout of groups, i.e., the functor $\pi_1 : \mathrm{Top} \to \mathrm{Grp}$ also preserves pushouts.

$$\begin{tikzcd}
\langle g \rangle \cong \mathbb{Z} & & 1 \\[-2em]
\pi_1(A, x_0) \arrow[rr, "(f_1)_*"] \arrow[d, "(f_2)_*"'] & & \pi_1(D^2, x_0) \arrow[d, "(g_1)_*"] \\
\pi_1(S^1 \vee S^1, x_0) \arrow[rr, "(g_2)_*"] & & \pi_1(T^2, x_0) \\[-2em]
\langle a, b \mid \ \rangle \cong \mathbb{Z} * \mathbb{Z} & & \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle
\end{tikzcd}$$

loop .... which is trivial in D^2

$$\begin{tikzcd}
g \arrow[r, mapsto] \arrow[d, mapsto] & 1 \arrow[d, mapsto] \\
aba^{-1}b^{-1} \arrow[r, mapsto] & 1
\end{tikzcd}$$

path in T^2 ...... which is trivial in T^2