# 6. Nilpotent and finite abelian groups

Big idea: The derived series of a group starts at the top of the subgroup lattice and takes "maximal abelian steps" down.

⋆ In this section, we'll start from the bottom and work our way up. Note that maximal abelian subgroups need not be unique. But every group has a center.

Process: Set $Z_0 = 1$

Set $Z_1 = Z(G)$. Now "discard everything below $Z_1$" (i.e., take $G/Z_1$)

Let $Z_2$ be s.t. $Z_2/Z_1 = Z(G/Z_1)$

Now, "discard everything below $Z_2$" (i.e, take $G/Z_2$)

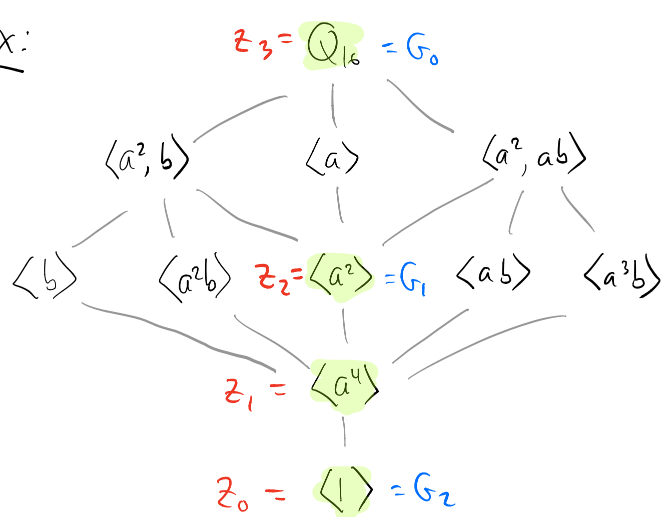
Let $Z_3$ be s.t. $Z_3/Z_2 = Z(G/Z_2)$

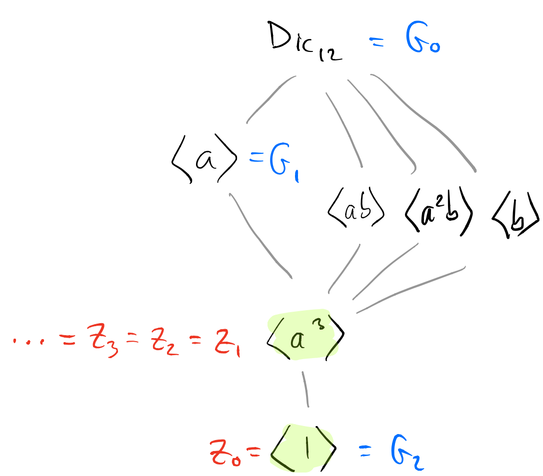
Repeat this process, and we get the ascending central series of $G$:

$$1 = Z_0 \leq Z_1 = Z(G) \leq Z_2 \leq Z_3 \leq \cdots$$

Def: $G$ is nilpotent if $G = Z_n$ for some $n$, and of class $n$ if $n$ is minimal.

Ex:



Remarks: • Solvable and nilpotent groups are both generalizations of abelian groups.

• If $n \geq 3$, then $Z(S_n) = 1$, thus $S_n$ is not nilpotent.

Prop 6.1: p-groups are nilpotent.

Pf: Suppose $|G| = p^n$. Since $Z(G) \gneq 1$, $G/Z_1$ is a p-group, so $Z_2/Z_1 = Z(G/Z_1) \neq 1$. Thus $Z_2 \gneq Z_1$. Likewise, if $Z_2 \neq G$, then $Z_3 \gneq Z_2$, and so on.

Prop 6.2: If $G$ is nilpotent, then it is solvable.

Pf: $Z_{i+1}/Z_i$ is abelian (it is $Z(G/Z_i)$), thus $G = Z_n \geq Z_{n-1} \geq \cdots \geq Z_0$ is a subnormal series with abelian factors. $\square$

Cor: p-groups are solvable.

Remark: $Z_{i+1}/Z_i = Z(G/Z_i) \Rightarrow Z_{i+1} = \{x \in G \mid xZ_i yZ_i = yZ_i xZ_i \ \forall y \in G\}$

$$= \{x \in G \mid [x,y] \in Z_i \ \forall y \in G\} \quad (\star)$$

Prop 6.3: If $G$ is nilpotent and $H \lneq G$, then $N_G(H) \gneq H$.

$$\begin{tikzcd} G \arrow[d, dash] & G/Z_i \arrow[d, dash] \\ Z_{i+1} \arrow[d, dash] & Z_{i+1}/Z_i \arrow[d, dash] \\ Z_i & Z_i/Z_i \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[d, dash] \arrow[dr, dash] & \\ Z_{i+1} \arrow[d, dash] & H \arrow[dl, dash] \\ Z_i \arrow[d, dash] & \\ Z_0 = 1 & \end{tikzcd}$$

Pf: For some $i$, $Z_i \leq H$ and $Z_{i+1} \nleq H$.

We'll show $Z_{i+1} \leq N_G(H)$.

Pick $x \in Z_{i+1}$.

By $(\star)$, $xhx^{-1}h^{-1} \in Z_i \leq H \ \forall h \in H$

$\Rightarrow xhx^{-1} \in H \ \forall h \in H \Rightarrow x \in N_G(H)$.

Cor: If $G$ is nilpotent and $H \lneq G$ maximal, then $H \trianglelefteq G$.

The converse holds as well if $G$ is finite (HW exercise).

Non-example: $\langle (12) \rangle \leq S_3$ maximal but $\langle (12) \rangle \not\trianglelefteq S_3$.

Prop 6.4: For a Sylow $p$-subgroup $P$, $N_G(N_G(P)) = N_G(P)$

Proof: Pick $x \in N_G(N_G(P))$. Then $x N_G(P) x^{-1} \subseteq N_G(P) \Rightarrow x P x^{-1} \leq N_G(P)$.

But $P \triangleleft N_G(P) \iff P$ is the unique Sylow $p$-subgroup of $N_G(P)$

$\iff x P x^{-1} = P$ $\square$

Def: If $H, K \leq G$, define $[H,K] = \langle [h,k] : h \in H, k \in K \rangle$.

Remarks:
- The $k^{th}$ commutator subgroup is $G^{(k)} = [G^{(k-1)}, G^{(k-1)}]$
- $H \trianglelefteq G$ iff $[G,H] \leq H$.

Prop 6.5: If $K \trianglelefteq G$ and $K \leq H \leq G$, then $H/K \leq Z(G/K)$ iff $[G,H] \leq K$.

Pf: $[g,h] = g^{-1}h^{-1}gh \in K \quad \forall g \in G, h \in H$

$\Updownarrow$

$ghK = hgK \quad \forall g \in G, h \in H$

$\Updownarrow$

$H/K \leq Z(G/K)$

$$\begin{tikzcd}[arrows=dash]
G \arrow[dd] \arrow[dr] & \\
& Z \\
H \arrow[ur] \arrow[d] & \\
K &
\end{tikzcd}
\qquad
\begin{tikzcd}[arrows=dash]
G/K \arrow[dd] \arrow[dr] & \\
& Z(G/K) \\
H/K \arrow[ur] \arrow[d] & \\
K/K &
\end{tikzcd}$$

$\square$

Note how this generalizes Property (★) above.

Def: The descending central series of $G$ is the series

$$G = L_0 \geq L_1 \geq L_2 \geq \cdots, \quad \text{where}$$

$$L_1 = [G,G], \quad L_2 = [G, L_1], \ldots, L_{k+1} = [G, L_k].$$

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**Remarks:** • $L_k \trianglelefteq G$ for each $k$. (since inverses & conjugates of commutators are commutators.)

• By Prop 6.5: $\boxed{\begin{array}{l} L_{k+1} = [G, L_k] \implies L_k/L_{k+1} \leq Z(G/L_{k+1}) \\ Z_k \geq [G, Z_{k+1}] \impliedby Z_{k+1}/Z_k = Z(G/Z_k) \end{array}}$

$$\begin{tikzcd}[column sep=small]
G = L_0 \arrow[dd, dash] \arrow[dr, dash] & \\
& Z \arrow[dl, dash] \\
L_k \arrow[d, dash] & \\
L_{k+1} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
G/L_{k+1} \arrow[dd, dash] \arrow[dr, dash] & \\
& Z(G/L_{k+1}) \arrow[dl, dash] \\
L_k/L_{k+1} \arrow[d, dash] & \\
L_{k+1}/L_{k+1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[d, dash] \\
Z_{k+1} \arrow[d, dash] \\
Z_k
\end{tikzcd}
\qquad
\begin{tikzcd}
G/Z_k \arrow[d, dash] \\
Z_{k+1}/Z_k = Z(G/Z_k) \arrow[d, dash] \\
Z_k/Z_k
\end{tikzcd}$$

**Lemma:** (HW). Suppose $N, H, K$ are normal with $N \leq H \leq G$.
If $[G, H] \leq N$ and $[G, K] \leq N$, then $[G, HK] \leq N$.

$$\begin{tikzcd}[column sep=small]
& G \arrow[d, dash] & \\
& HK \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
& N &
\end{tikzcd}$$

**Thm 6.6:** $G$ is nilpotent iff $L_n(G) = 1$ for some $n$.

**Pf:** ($\Rightarrow$) Suppose $G$ is nilpotent of class $n$. Then

$$L_1 = [G, L_0] = [G, Z_n] \leq Z_{n-1}$$
$$L_2 = [G, L_1] \leq [G, Z_{n-1}] \leq Z_{n-2}$$
$$\vdots$$
$$L_k = [G, L_{k-1}] \leq [G, Z_{n-k+1}] \leq Z_{n-k}$$

$$\begin{tikzcd}[column sep=small]
L_0 = Z_n = G \arrow[d, dash] & \\
L_1 \arrow[d, dash] \arrow[r, dash] & Z_{n-1} \arrow[d, dash] \\
L_2 \arrow[d, dash, dotted] \arrow[r, dash] & Z_{n-2} \arrow[d, dash, dotted] \\
{} & {}
\end{tikzcd}$$

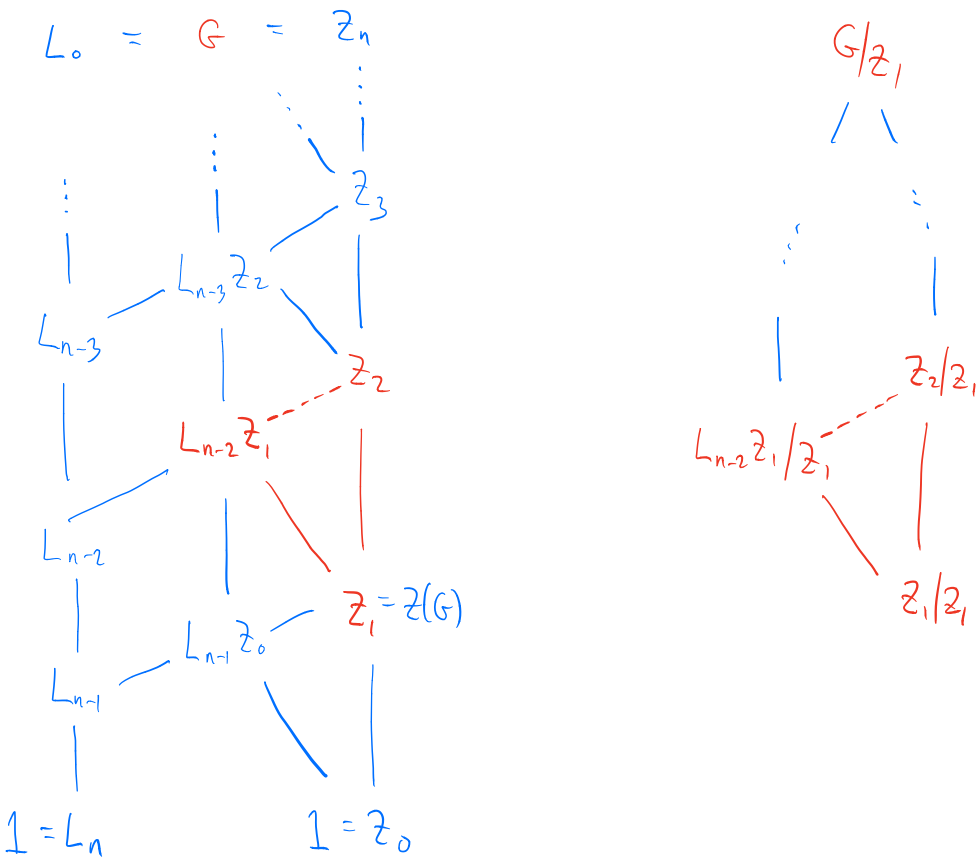
Note: The middle inequality follows from $K \leq H \Rightarrow [G, K] \leq [G, H]$.

Since $G$ is nilpotent, $L_n \leq Z_0 = 1$. ✓

($\Leftarrow$) Suppose $L_n = 1$ for some (minimal) $n$.

$[G, L_{n-1}] = L_n = 1 \Rightarrow L_{n-1} \leq Z_1 = Z(G)$ (Prop 6.5)

$[G, L_{n-2}] = L_{n-1} \leq Z_1$. We'll show $L_{n-2} \leq Z_2$, and so on.




Consider $Z_1 \leq L_{n-2}Z_1 \leq G$. [Goal: show $L_{n-2}Z_1 \leq Z_2$.]

$[G, L_{n-2}] = L_{n-1} \leq Z_1$ and $[G, Z_1] \leq Z_0 \leq Z_1$

By Lemma, $[G, L_{n-2}Z_1] \leq Z_1$

Prop 6.5 $\Rightarrow$ $L_{n-2}Z_1/Z_1 \leq Z(G/Z_1) := Z_2/Z_1$.

Thus, $L_{n-2} \leq L_{n-2}Z_1 \leq Z_2$.

Similarly, $L_{n-3} \leq L_{n-3}Z_2 \leq Z_3$

$\vdots$

$G = L_0 \leq L_0 Z_n \leq Z_n = G$. $\square$

<u>Prop 6.7</u>: If $H, K$ are nilpotent, then $G = H \times K$ is nilpotent.

<u>Pf:</u> Clearly, $L_0(G) = L_0(H) \times L_0(K)$.

Suppose $L_k(G) = L_k(H) \times L_k(K)$.

Then $L_{k+1}(G) = [H \times K, L_k(H) \times L_k(K)] = [H, L_k(H)] \times [K, L_k(K)]$

$$= L_{k-1}(H) \times L_{k-1}(K).$$

If $H$ and $K$ are nilpotent of class $m$ and $n$, resp., then $L_N(G) = 1$ if $N \geqslant m, n$. Thus $G$ is nilpotent. $\square$

<u>Prop 6.8</u>: A finite group $G$ is nilpotent iff it is the internal direct product of its Sylow subgroups.

<u>Pf</u>: $(\Longleftarrow)$ Easy.

$(\Longrightarrow)$ Let $G$ be nilpotent and $P \leq G$ be $p$-Sylow.

Then $N_G(N_G(P)) = N_P(G)$.

By Prop 6.3, $N_G(N_G(P)) \gneqq N_G(P)$, unless $N_G(P) = G$.

Thus, $N_G(P) = G \Rightarrow P \trianglelefteq G$.

Now, let $P_1, \dots, P_n$ be distinct nontrivial Sylow subgroups of $G$.

(i) $G = P_1 = P_1 P_2 \cdots P_n$. ✓

(ii) $P_i \lhd G \quad \forall i$. ✓

(iii) $P_i \cap \langle \bigcup P_j : j \neq i \rangle = 1$ ?

<u>Check:</u> If $x \in P_i$, $y \in P_j$, then $xyx^{-1}y^{-1} \in P_i \cap P_j = 1 \Rightarrow xy = yx$.

If $x \neq 1$, then $|x| \nmid \langle \bigcup P_j : j \neq i \rangle$, so $x \notin \langle \bigcup P_j : j \neq i \rangle$.

By Thm 5.3, $G \cong P_1 \times \cdots \times P_n$. $\square$

## Summary of nilpotent groups

A finite group $G$ is nilpotent if one of the following conditions hold:

(i) $Z_n = G$ for some $n$ in the ascending central series.

(ii) $L_n = 1$ for some $n$ in the descending central series.

(iii) For each $H \lneq G$, $H \lneq N_G(H)$.

(iv) Every Sylow subgroup of $G$ is normal

(v) $G \cong P_1 \times \cdots \times P_r$ (its Sylow subgroups).

(vi) Every maximal proper subgroup is normal (see HW).

## Finite abelian groups

Thm 6.9: If $G$ is a finite abelian group, then $G$ is a direct sum of cyclic subgroups, each of prime power order.

Pf: Since $G$ is abelian, it is nilpotent, thus is the direct product of its Sylow subgroups.

Thus, we may assume $|G| = p^n$.

Induct on $n$. Base case is trivial.

Pick $a \in G$ of maximal order, say $|a| = p^k$, and choose $H \leq G$ maximal w.r.t. $H \cap \langle a \rangle = 0$.

Set $G_1 = H \oplus \langle a \rangle \leq G$.

Claim: $G_1 = G$. [Then we can apply the IHOP and be done.]

If not, then pick an element $x + G_1 \in G/G_1$ of order $p$, i.e., $px \in G_1$.

Say $px = h + ma \in H \oplus \langle a \rangle$.

Since $|x| \leq p^k$, $0 = p^k x = p^{k-1}(px) = p^{k-1}h + p^{k-1}ma.$ $(*)$

Note: $p^{k-1}h \in H$ and $p^{k-1}ma \in \langle a \rangle$ are inverses, and thus in $H \cap \langle a \rangle = 0$.

Now, $p^{k-1}ma = 0 \implies p \mid m$ (recall: $|a| = p^k$.)

So write $m = pr$, where $r \in \mathbb{Z}$.

From $(*)$, we have $p^{k-1}h = p^k x - p^{k-1}ma = p^k x - p^k ra$

Solving for $h$: $h = p(x - ra) \in H$

But note that $ra \in G_1$ and $x \notin H \leq G_1$ $\implies x - ra \notin H$.

By maximality of $H$, $(H + \langle x - ra \rangle) \cap \langle a \rangle \neq 0$

So, pick some $h_1 + t(x - ra) = sa \neq 0$ in it.

$$\implies tx = -h_1 + (s + tr)a \in H \oplus \langle a \rangle = G_1$$

By choice of $x$, $tx \in G_1 \implies p \mid t \implies t = up$ for some $u \in \mathbb{Z}$.

But now we'll show no such $x$ can exist:

$$0 \neq \underbrace{sa}_{\text{in } \langle a \rangle} = h_1 + up(x - ra) = \underbrace{h_1 + uh}_{\text{in } H} \in H \cap \langle a \rangle = 0. \ \lightning$$

$\square$

Thm 6.10: Suppose $G$ is abelian, $|G| = p^n$ for some prime $p$, and $G = G_1 \oplus \cdots \oplus G_r = H_1 \oplus \cdots \oplus H_s$, with each $G_i$, $H_j$ cyclic and $1 < |G_i| < |G_{i+1}|$, $1 < |H_j| \leq |H_{j+1}|$.

Then $r = s$ and $G_i \cong H_i$ for $1 \leq i \leq r$.

Pf: Let $H = \{x \in G \mid px\} = 0$.

Each cyclic subgroup has a unique subgroup $K_i \leq G_i$ of order $p$, so $H = K_1 \oplus K_2 \oplus \cdots \oplus K_r$ and $|H| = p^r$.

Similarly, $|H| = p^s \implies r = s$.

Suppose now that $k$ is the minimal index s.t. (wlog) $|H_k| > |G_k| = p^t$.

Then $p^t G = p^t G_{k+1} \oplus \cdots \oplus p^t G_r = p^t H_k \oplus \cdots \oplus p^t H_r$.

This contradicts what we just proved: there must be the same number of factors. $\square$

<u>Cor</u>: (Fundamental theorem of finite abelian groups.) Suppose $G$ is a finite abelian group, with primes $p_1 < \cdots < p_k$ dividing $|G|$. Then $G = G_1 \oplus \cdots \oplus G_k$, where $G_i$ is a $p_i$-group, and $G_i = H_{i,1} \oplus \cdots \oplus H_{i,m_i}$ for each $i$, with $H_{i,j}$ cyclic and $1 < |H_{i,j}| \leq |H_{i,j+1}|$ for each $j$.

<u>Pf:</u> Each $G_i$ is a Sylow $p_i$-subgroup, and the second part follows from Theorems 6.9 and 6.10. $\square$