7. Free groups, free objects, and group presentations

Goal of this section: We've seen group presentations like

(i) $D_4 = \langle r, f \mid r^4 = f^2 = 1,\ rfr = r \rangle$.

(ii) $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle$

(iii) $\mathbb{Z}_5 = \langle x \mid x^5 = 1 \rangle$.

But what does this really mean?

For example, take $G = \{1\}$, $r = f = 1$. Then r, f certainly "satisfy" the presentation in (i). Same for $G = \{1, -1\} \cong \mathbb{Z}_2$.

To formalize the notion of a group presentation, we'll need to introduce free groups, and for that we'll need free semigroups.

Def: A semigroup is a nonempty set with an associative binary operation (think: "group, with identity & inverses optional")

A semigroup homomorphism is a function $f: X \to Y$ with $f(x_1 x_2) = f(x_1) f(x_2)$ for all $x_1, x_2 \in X$. If f is bijective, then $X \cong Y$.

Def: A semigroup X is free on a set S if there is a function $j: S \to X$ s.t. for any other function $k: S \to Y$ to a semigroup, $\exists!$ homom. $f: X \to Y$ s.t. $fj = k$.

$$\begin{tikzcd} S \arrow[r, "j"] \arrow[dr, "k"'] & X \arrow[d, dashed, "f"] \\ & Y \end{tikzcd}$$

Example: Let $S = \{1\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$. Then $(\mathbb{N}, +)$ is a free semigroup on S with $j(1) = 1$. (Verify on HW.)

This means that there <u>every</u> semigroup (or group) on one generator is a quotient of $\mathbb{N}$, e.g.,

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[dr] & \mathbb{N} \arrow[d, dashed] \\ & \mathbb{Z}_4 \end{tikzcd} \qquad \begin{tikzcd} \{1\} \arrow[r] \arrow[dr] & \mathbb{N} \arrow[d, dashed] \\ & \mathbb{Z} \end{tikzcd} \qquad \begin{tikzcd} \{1\} \arrow[r] \arrow[dr] & \mathbb{N} \arrow[d, dashed] \\ & (\mathbb{Z}, \cdot) \end{tikzcd}$$

<u>Prop 7.1</u>: If a free semigroup exists on S, it is unique up to isomorphism.

<u>Pf</u>: HW.

<u>Thm 7.2</u>: If $S \neq \emptyset$, then there exists a free semigroup on S.

<u>Pf</u>: We will construct it explicitly.

Set $X = S \cup (S \times S) \cup (S \times S \times S) \cup \cdots$ "all finite words over S"

Define a binary operation of concatenation:

$$(a_1, \ldots, a_m)(b_1, \ldots, b_k) = (a_1, \ldots, a_m, b_1, \ldots, b_k).$$

This is associative, thus X is a semigroup.

Let $j : S \to X$ be the inclusion map: $j(x) = x$.

<u>Claim</u>: X is free on S.

To show this, let $k : S \to Y$ be a function to a semigroup.

Define $f : X \to Y$ by $f((a_1, \ldots, a_m)) = k(a_1) \cdots k(a_m)$.

<u>Check</u>: f is a homom & $fj = k$ ✓

<u>Uniqueness</u>: Say $g: X \to Y$ also satisfies $fj = gj = k$.

$$\begin{tikzcd} S \arrow[r, hook, "j"] \arrow[dr, "k"'] & X \arrow[d, dashed, shift right, "g"'] \arrow[d, dashed, shift left, "f"] \\ & Y \end{tikzcd}$$

Then
$$\begin{aligned} g((a_1,\dots,a_m)) &= g((ja_1,\dots,ja_m)) \\ &= g(ja_1)\cdots g(ja_m) \\ &= k(a_1)\cdots k(a_m) \\ &= f(ja_1)\cdots f(ja_m) = f((a_1,\dots,a_m)) \Rightarrow f = g. \checkmark \end{aligned}$$

Since $\exists!\ f$ s.t. $fj = k$, X is a free semigroup on S. $\square$

<u>Prop 7.3</u>: (Quotient semigroups & their universal property)

Let Y be a semigroup and R an equivalence relation satisfying

xRy & $zRw \Rightarrow xzRyw$ (i.e., it is well-defined w.r.t. the operation.)

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[dr, "h"'] & Y/R \arrow[d, dashed, "f"] \\ & Z \end{tikzcd}$$

Then Y/R is a semigroup if $\bar{x}\,\bar{y} := \overline{xy}$

and it satisfies the following <u>universal property</u>:

"If $h: Y \to Z$ is a semigroup homomorphism

with $fq = h$ and satisfying $h(x) = h(z)$ whenever xRy."

In other words: "every homomorphism respecting R factors through Y/R."

<u>Pf:</u> Existence: $f([x]) := f(q(x)) = h(x)$. ✓

Uniqueness: If $fq = gq$, then $f = g$ because q is an epi. $\square$

<u>Def</u>: A group F is free on a nonempty set S if $\exists$ function $\phi: S \to F$ s.t. for any other function $\theta: S \to G$ to a group, $\exists!$ homom. $f: F \to G$ s.t. $f\phi = \theta$.

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[dr, "\theta"'] & F \arrow[d, dashed, "f"] \\ & G \end{tikzcd}$$

<u>Goal:</u> We'll show that free groups (if they exist) are unique up to isomorphism, and then we'll show they exist by constructing them as quotients of free semigroups.

<u>Prop 7.4:</u> (Uniqueness.) If a free group exists on $S \neq \emptyset$, it is unique up to isomorphism, and ϕ is 1-1.

<u>Pf:</u> <u>Uniqueness:</u> Exercise (HW)

<u>Injectivity of ϕ:</u> Suppose ϕ were not 1-1, and $a \neq b$ but $\phi(a) = \phi(b)$.

Define $\Theta : S \longrightarrow \mathbb{Z}$

$$\begin{aligned} a &\longmapsto 1 \\ b &\longmapsto 2 \\ c &\longmapsto 0 \qquad c \neq a, b \end{aligned}$$

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[dr, "\theta"'] & F \arrow[d, dashed, "f?"] \\ & \mathbb{Z} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[rr, mapsto] \arrow[dd] & & x \arrow[ddl, dashed, "\nexists f"] \\ & b \arrow[ur, mapsto] \arrow[d] & \\ 1 & 2 & \end{tikzcd}$$

Then $1 = \theta(a) = f\phi(a) = f\phi(b) = \theta(b) = 2$.

Thus, no such $f : F \to \mathbb{Z}$ can exist with $f\phi = \theta$, so F is not free. $\square$

<u>Thm 7.5:</u> (Existence). If $S \neq \phi$, then there is a free group F on S.

<u>Pf:</u> Choose a set S' with $|S| = |S'|$, $S \cap S' = \phi$, and put $T = S \cup S'$.

(think of S' as the "inverses" of elts in S)

Let $s \longmapsto s'$ be a 1-1 correspondence b/w S and S', and

$s' \longmapsto (s')' = s'' = s$ the inverse map $S' \to S$.

Thus, $t \mapsto t'$ is a bijection $T \to T$.

Let X be the free semigroup on T (exists by Thm. 7.2).

If $g : X \to G$ is a homom. to a group, call g <u>proper</u> if $g(s') = g(s)^{-1}$ for all $s \in S$.

It follows easily that $g(t') = g(t)^{-1}$ for all $t \in T$.

☆ Define a relation R on X where:

$x R y$ iff "$g(x) = g(y)$ holds whenever $g: X \to G$ is proper."

<u>Motivation:</u> $g(xyy^{-1}) = g(x)$ so $xyy^{-1} R x$.

<u>Check:</u>
- R is an equiv. relation on X
- xRy and $zRw \implies xzRyw$.

Therefore, $F = X/R$ is a semigroup and $q: X \longrightarrow X/R$ is a homom. (Prop 7.3). Write $\bar{x} = q(x)$.

Note that $\overline{xy} = \bar{x}\,\bar{y}$ and $\overline{x'} = \bar{x}^{-1}$.

<u>Claim:</u> (i) F is a group

(ii) F is free on S.

<u>Pf of claim:</u>

(i) <u>F is a group</u>

Choose $s \in S$, $x \in X$ (under natural inclusion, $s, s' \in X$).

<u>Identity:</u> We'll show $\overline{ss'} \in X/R$ is the identity elt.

If $g: X \to G$ is proper, $g(ss') = 1$, and so $g(ss'x) = g(x)$.

By definition of R, $xRxss' \iff \bar{x} = \overline{xss'} = \bar{x}\,\overline{ss'}$

Similarly, $\overline{ss'}\,\bar{x} = \bar{x}$, so $\overline{ss'} = 1_F$.

<u>Inverses:</u> let $x = t_1 t_2 \cdots t_k$, $t_i \in T$.

Then $(\bar{x})^{-1} = \bar{y}$, where $y = t_k' \cdots t_2' t_1'$.

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<u>Check:</u> We want to show $\overline{x}\,\overline{y} = 1_F = \overline{ss'}$ for any $s \in S$,

or equivalently, $xy\,R\,ss'$:

If $g: X \to G$ is proper, then

$$\begin{aligned} g(xy) &= g(s_1 s_2 \dots s_k s_k' \dots s_2' s_1') \\ &= g(s_1) g(s_2) \cdots g(s_k) g(s_k') \cdots g(s_2') g(s_1') \\ &= g(s_1) g(s_2) \cdots g(s_k) g(s_k)^{-1} \cdots g(s_2)^{-1} g(s_1)^{-1} \\ &= 1_G = g(ss') \text{ for any } s \in S. \quad \checkmark \end{aligned}$$

Since $F = X/R$ is a semigroup with identity & inverses, it is a group.

☆ Think of F as the group of finite words over $S \cup S^{-1}$ where the binary operation is concatenation.

(ii) <u>F is free on S</u>:

Let $S \xhookrightarrow{i} T \hookrightarrow X$ be inclusion maps.

Define $\phi: S \to F$ by $\phi = q j i$.

Now, let $\theta: S \to G$ be any function to a group.

Extend θ to $\theta': T \to G$ by setting $\theta'(s') = \theta(s)^{-1}$ $\forall s \in S$.

☆ <u>Goal</u>: Show $\exists!$ homom. $f: X \to G$ s.t. $f\phi = \theta$.

$$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[rrr, bend left, "\phi"] \arrow[dr, "\theta"'] & T \arrow[r, hook, "j"] \arrow[d, "\theta'"] & X \arrow[r, two heads, "q"] \arrow[dl, dashed, "\exists! \tau"] & F = X/R \arrow[dll, dashed, "\exists! f"] \\ & G & & \end{tikzcd}$$

Prop 7.2 (X is free over T)

Prop 7.3 (Universal prop.)

Since X is free on T, $\exists!\ \gamma : X \to G$ s.t. $\gamma j = \theta'$. (Prop 7.2)

By Prop 7.3 (Univ. property), $\exists!\ f : F \to G$ s.t. $fq = \gamma$.

<u>Note:</u> $f\phi = fqji = \gamma ji = \theta' i = \theta$.

<u>Uniqueness of f:</u> Suppose $\exists\, h : F \to G$ s.t. $h\phi = \theta$.

☆ If we can show $hq = fq = \gamma$, then uniqueness of h will follow from Prop 7.3.

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\gamma"'] & X/R \arrow[dl, dashed, "h"'] \arrow[dl, dashed, bend left, "f"] \\ G & \end{tikzcd}$$

<u>Note:</u> $hqji = \theta = \theta' i = \gamma ij = fqji$.

Thus $hq(s) = fq(s)$ $(= \gamma(s))$ $\quad \forall\, s \in S \subseteq X$.

Need to show $hq(s') = fq(s')$ $(= \gamma(s'))$ $\quad \forall\, s' \in S' \subseteq X$.

Recall that $q(s') = q(s)^{-1}$ in $F = X/R$.

Thus $h(q(s')) = h(q(s)^{-1}) = (hq(s))^{-1} = (fq(s))^{-1} = f(q(s)^{-1}) = f(q(s')).$ ✓

$\square$

Since ϕ is 1-1, we identify $s \in S$ with $\phi(s) \in F$, and just say $S \subseteq F$.

The elements of S are generators of F, and we write $F = F_S = \langle S \rangle$.

<u>Remark:</u> By definition, any function $\theta : S \to G$ (arbitrary G) can be extended <u>uniquely</u> to a homom. $f : F_S \to G$ s.t. $f\phi = \theta$.

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "\theta"'] & & F_S \arrow[dl, dashed, "f"] \\ & G & \end{tikzcd}$$

<u>Examples:</u>

(1) $|S| = 1$, say $S = \{s\}$ and let $\phi : S \hookrightarrow \mathbb{Z}$
$$s \longmapsto 1$$

let $\theta : S \to G$ be <u>any</u> map; say $\theta(s) = g$.

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Then the homom. $f : \mathbb{Z} \longrightarrow G$
$\quad\quad 1 \longmapsto g$

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\theta"'] & \mathbb{Z} \arrow[dl, dashed, "f"] \\ G & \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "\theta"'] & 1 \arrow[dl, dashed, mapsto, "f"] \\ g & \end{tikzcd}$$

is the unique homom. s.t. $f\phi = \theta$.

Thus, $\mathbb{Z}$ is free on S.

Note that $\langle g \rangle \cong \mathbb{Z}$ or $\mathbb{Z}_n$, thus every cyclic group is the quotient of the free group on $\{s\}$ (one generator).

(2) Let $S = \{a, b\}$.

$$\begin{tikzcd} \{a, b\} \arrow[r, hook, "\phi"] \arrow[d, "\theta"'] & F_S \arrow[dl, dashed, "f"] \\ \langle (12), (123) \rangle & \end{tikzcd}$$

Note that S_3 is not cyclic; it has two generators: $S_3 = \langle (12), (123) \rangle$.

Let $\phi : \{a, b\} \hookrightarrow F_S$.

Define $\theta : \{a, b\} \longrightarrow S_3 = \langle (12), (123) \rangle$
$\quad a \longmapsto (12)$
$\quad b \longmapsto (123)$.

The free group F_S is the set of all words over $S = \{a, b\}$ under concatenation, which we write as $\langle a, b \mid \ \rangle$; i.e., 2 generators, no relations.

The map $\theta : S \longrightarrow S_3$ extends to a unique homom. $f : F_S \rightarrow S_3$.

Big idea: The group S_3 is generated by 2 elements, and is a quotient of the free group F_S, where $|S| = 2$.

☆ More generally, if $G = \langle S \rangle$ and $|S| = n$, then $\exists$ homom. $F_S \twoheadrightarrow G$, i.e., every group is the quotient of a free group.

<u>Why</u>:
$$\begin{tikzcd} S \arrow[r, hook, "\phi"] \arrow[dr, "\theta"'] & F_S \arrow[d, dashed, "f"] \\ & \langle G \rangle \end{tikzcd}$$
and f is surjective because $\langle S \rangle = G$.

<u>Thm 7.6</u>: Suppose $S, U \neq \emptyset$. Then $F_S \cong F_U$ iff $|S| = |U|$.

<u>Pf</u>: ($\Longrightarrow$) <u>Case 1</u>: $|S| < \infty$

$$\begin{tikzcd} S \arrow[r, hook, "\phi"] \arrow[dr, "\theta"'] & F_S \arrow[d, dashed, "f"] \\ & \mathbb{Z}_2 \end{tikzcd}$$

Since $F_S \cong F_U$, they have the same number of index-2 subgroups.

Each is characterized uniquely as the kernel of a surjection $f: F_S \longrightarrow \mathbb{Z}_2$.

Clearly, there are $2^{|S|} - 1$ such surjections.

So F_S has $2^{|S|}-1$ index-2 subgroups, and F_U has $2^{|U|}-1 \Rightarrow |S| = |U|$. ✓

<u>Case 2</u>: $|S| = \infty$. Set $T = S \cup S^{-1}$, so $|T| = |S|$.

$$|F_S| \leq 1 + |T| + |T \times T| + |T \times T \times T| + \dots = \aleph_0 |T| = |S|.$$

Therefore, $|F_S| = |S|$, and so $|S| = |F_S| = |F_U| = |U|$. ✓

($\Longleftarrow$) Suppose $h: S \longrightarrow U$ is bijective,

$$\begin{tikzcd} S \arrow[r, hook, "\phi_S"] \arrow[d, "h"'] \arrow[dr, "\phi_U h"] & F_S \arrow[d, dashed, "f"] \\ U \arrow[r, hook, "\phi_U"] \arrow[d, "h^{-1}"'] \arrow[dr, "\phi_S h^{-1}"] & F_U \arrow[d, dashed, "g"] \\ S \arrow[r, hook, "\phi_S"'] & F_S \end{tikzcd}$$

$\phi_S: S \hookrightarrow F_S$, $\phi_U: U \hookrightarrow F_U$.

Then $\phi_U h: S \hookrightarrow F_U$, so $\exists!\, f: F_S \to F_U$ s.t. $f\phi_S = \phi_U h$.

Similarly, $\phi_S h^{-1}: U \hookrightarrow F_S$, so $\exists!\, g: F_U \to F_S$ s.t. $g\phi_U = \phi_S h^{-1}$.

Now, $gf: F_S \longrightarrow F_S$ satisfies $\phi_S = gf\phi_S = 1_S \phi_S$.

By uniqueness, $gf = 1_{F_S}$.

Similarly, $fg = 1_{F_U}$, so f & g are inverse isomorphisms, and $F_S \cong F_U$. $\square$

Def: The rank of a free group F_S is $|S|$.

Thm: Subgroups of free groups are free.

Thm: If $1 < |S|, |U| \leq \aleph_0$, then $\exists$ embedding $F_S \hookrightarrow F_U$.

Proofs: Involve algebraic topology (covering spaces).

Not surprisingly, the concept of a "free" object can be defined in a categorical setting.

Def: A concrete category is a category $\mathcal{C}$ where the objects $A \in \mathrm{Ob}(\mathcal{C})$ have an underlying set structure $\mathcal{F}(A)$, and

(i) Every $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is a function on the underlying sets:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(B) \end{tikzcd}$$

(ii) The identity morphism is the identity function on the sets:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d] & A \arrow[d] \\ \mathcal{F}(A) \arrow[r, "1_{\mathcal{F}(A)}"] & \mathcal{F}(B) \end{tikzcd}$$

(iii) Composition of functions agrees with composition of functions on the sets.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[d] \\ \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(B) \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(C) \end{tikzcd}$$

Formally, this means that there is a covariant functor $\mathcal{F}$ to the category **Set**.

Informally: Objects are sets with extra structure (e.g., groups)

Morphisms are functions with extra structure (e.g., homomorphisms).

Non-examples:

(1) Any directed graph defines a non-concrete category, assuming loops & transitive edges.

(2) Let G be a fixed group. $Ob(G) := \{G\}$ (i.e., one object).

$Hom(G) := \{g : g \in G\}$. The identity element is 1_G.

Def: Let F be an object in a concrete category $\mathcal{C}$, $S \neq \emptyset$, and $\phi : S \longrightarrow F$ a map of sets. Then F is free on S if for any $A \in Ob(\mathcal{C})$ and set map $\theta : S \longrightarrow A$, $\exists!\, f \in Hom_{\mathcal{C}}(F, A)$ s.t. $f\phi = \theta$ (as maps of sets).

Thm 7.7: Let $F, F' \in Ob(\mathcal{C})$ be free objects on S, S', respectively, where $|S| = |S'|$. Then F and F' are equivalent.

Pf: HW.

Free objects are universal (i.e., initial) objects in an appropriately constructed category, like how co-products were.

<u>Construction</u>: (Details to be verified on HW).

Let $F \in Ob(\mathcal{C})$ be free on S, with $\phi: S \to F$.

Define a new category $\mathcal{D}$ as follows:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & A \arrow[d, dashed, "h"] \\ & B \end{tikzcd}$$

$Ob(\mathcal{D})$: maps of sets $S \xrightarrow{f} A \in Ob(\mathcal{C})$

$Hom(\mathcal{D})$: $h \in Hom_{\mathcal{D}}(S \xrightarrow{f} A,\ S \xrightarrow{g} B)$ if $h \in Hom_{\mathcal{C}}(A, B)$ and $hf = g$.

<u>Claim</u>:

- $Id_A : A \to A$ is the identity morphism on $S \xrightarrow{f} A$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & A \arrow[d, dashed, "Id_A"] \\ & A \end{tikzcd}$$

- $h \in Hom_{\mathcal{D}}(S \xrightarrow{f} A,\ S \xrightarrow{g} B)$ is an equivalence iff $h \in Hom_{\mathcal{C}}(A, B)$ is an equivalence.

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[dr, "\theta"'] & F \arrow[d, dashed, "f"] \\ & A \end{tikzcd}$$

- If F is free on S, then $S \xrightarrow{\phi} F$ is initial (i.e., universal) in $\mathcal{D}$.

Now, back to groups.

<u>Remark</u>: Every group G generated by S is a quotient of F_S.

By the FHT, $\langle S \rangle = F_S / K$, for some $K \trianglelefteq F_S$.

$$\begin{tikzcd} S \arrow[r, hook, "\phi"] \arrow[dr, "\theta"'] & F_S \arrow[d, dashed, "q"] \\ & \langle S \rangle \end{tikzcd}$$

Let $K = \langle R \rangle$. Then:

each $r \in R$ is an element of F_S,

i.e, a word in $S \cup S^{-1}$.

The quotient $q: F_S \to F_S / K$ maps $r \mapsto rK = K$; it "sets $r = 1$."

We say that G has a set S of <u>generators</u> G subject to a set of <u>relations</u> $\{r = 1 : r \in R\}$. The elements r are called <u>relators</u>.

Thus every group $G = \langle S \rangle$ is isomorphic to a quotient F_S/K, where $K := \bigcap_{R \subseteq N_\alpha \trianglelefteq F_S} N_\alpha$.

We write this as $G = \langle S \mid R \rangle := \langle S \mid r = 1 \ \forall r \in R \rangle$, called a presentation of G.

Ex: $C_n = \langle a \mid a^n = 1 \rangle$.

$\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$.

$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$.

Prop 7.8: Suppose $G_1 = \langle S \mid R_1 \rangle$, $G_2 = \langle S \mid R_2 \rangle$, and $R_1 \subseteq R_2$.

Then $\exists$ homom. $g : G_1 \twoheadrightarrow G_2$.

Pf: Assume $\pi : F_S \twoheadrightarrow F_S / K_1 \cong G_1$ is the canonical quotient map.

and $K_1 = \bigcap_{R \subseteq N_\alpha \trianglelefteq F_S} N_\alpha$.

$$\begin{tikzcd}
S \arrow[r, hook, "\phi"] \arrow[d, "\theta"'] & F_S \arrow[d, "\pi"] \arrow[dl, dashed, "f"'] \\
G_2 & G_1 \arrow[l, dashed, "g"]
\end{tikzcd}$$

Let $\phi : S \hookrightarrow F_S$ and $\theta : S \hookrightarrow G_2$ be inclusions.

Since F_S is free, $\exists!\ f : F_S \twoheadrightarrow G_2$ s.t. $\theta = f\phi$.

Since $R_1 \subseteq R_2$, $\ker \pi \leq \ker f$, so by the universal prop. of quotient groups, $\exists!$ homom. $g : G_1 \to G_2$ s.t. $f = g\pi$. $\square$

To summarize Prop 7.8:

- "Adding relations induces a homomorphism"
- $\langle S \mid R \rangle$ is the "freest group subject to the relations."

Note: Removing generator s_i is equivalent to adding the relation $s_i = 1$.

Thus, if $S_1 \supseteq S_2$ and $R_1 \subseteq R_2$, then $\exists$ homom. $\langle S_1 | R_1 \rangle \twoheadrightarrow \langle S_2, T_2 \rangle$.

Actually, we don't need $S_1 \supseteq S_2$, but rather just any surjection $\theta : S_1 \twoheadrightarrow S_2$ that "respects relations", i.e., $r = 1 \Rightarrow \theta(r) = 1$.

More precisely, we have established the following:

Cor 7.9: Suppose $G_1 = \langle S \mid R \rangle$ and $G_2 = \langle S' \mid R' \rangle$ such that

(i) $\exists \theta : S \twoheadrightarrow S'$; say $\theta(s) = s'$ and extending to $\theta : R \to R'$.

(ii) $r' = 1 \ \forall r' \in R$ (i.e., $\theta(r) = 1 \ \forall r \in R$).

Then $\exists$ homom. $g : G_1 \twoheadrightarrow G_2$.

Examples:

(1) Let $G_1 = \langle a \mid a^n = 1 \rangle$ and $G_2 = \langle b \mid b^m = 1 \rangle$, $m \mid n$.

Then $\theta : a \mapsto b$ and $\theta(a^n) = b^n = 1$, so θ extends to a homom.

$G_1 \twoheadrightarrow G_2$. (In fact, $\mathbb{Z}_n \twoheadrightarrow \mathbb{Z}_m$.)

(2) Let $G = \langle a, b \mid a^3 = 1,\ b^2 = 1,\ abab = 1 \rangle$.

Note: $\left.\begin{array}{l} a^3 = 1 \Rightarrow a^{-1} = a^2 \\ b^2 = 1 \Rightarrow b^{-1} = b \end{array}\right\} \Longrightarrow abab = 1 \Leftrightarrow ab = ba^{-1} = ba^2$.

Thus, every element can be written in the form $b^i a^j$, where $i = 0, 1$ and $j = 0, 1, 2 \Longrightarrow |G| \leq 6$.

So $G \cong 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_6$, or S_3.

which one is it?

Consider $\sigma = (123)$, $\tau = (12)$ in $S_3 = \langle \sigma, \tau \rangle$.

Let $\theta(a) = \sigma$ and $\theta(b) = \tau$.

Check: $\theta(a^3) = \sigma^3 = 1$, $\theta(b^2) = \tau^2 = 1$, $\theta(abab) = \sigma\tau\sigma\tau = 1$.

By Cor 7.9, $\exists\, g : G \twoheadrightarrow S_3$, and so $|G| \geq 6$.

Therefore, $G \cong S_3 = \langle a, b \mid a^3, b^2, abab \rangle$.

(3) Let $G = \langle x, y \mid xy = y^2x,\ yx = x^2y \rangle$

Note: $xy = y^2x \Rightarrow y^{-1}(xy) = yx = x^2y = x(xy) \Rightarrow x = y^{-1}$

Thus, $1 = xy = y^2x = y(yx) = y \Rightarrow y = 1 \Rightarrow x = 1$.

Hence, $G = 1$.

(4) Let $G = \langle a, b \mid a^n = b^2 = 1,\ ab = b^{-1}a \rangle$

Note that $a^{-1} = a^{n-1}$, $b^{-1} = b$, $ab = ba^{n-1}$.

Thus, every element can be written as a^ib^j, $0 \leq i \leq n$, $0 \leq j < 2$,

hence $|G| \leq 2n$.

To show $|G| \geq 2n$, we'll demonstrate $G \twoheadrightarrow D_n$.

Consider $D = \langle r, f \rangle$, where $r = 2\pi/n$ - rotation, f = reflection.

Define $\theta(a) = r$, $\theta(b) = f$. ✓

Show θ "respects relations": $r^n = f^2 = 1$, $rf = fr^{-1}$

Thus $\exists\, G \twoheadrightarrow D_n$, and since $|G| \leq |D_n|$, then $G \cong D_n$

Fun fact: Given two finitely presented groups $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ (i.e., $|S_i|, |R_i| < \infty$), determining whether $G_1 \cong G_2$ is, in general, computationally undecidable!