

## 7. Free groups, free objects, and group presentations

□

Goal of this section: We've seen group presentations like

$$(i) D_4 = \langle r, f \mid r^4 = f^2 = 1, rfr = r \rangle.$$

$$(ii) \mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle$$

$$(iii) \mathbb{Z}_5 = \langle x \mid x^5 = 1 \rangle.$$

But what does this really mean?

For example, take  $G = \{1\}$ ,  $r = f = 1$ . Then  $r, f$  certainly "satisfy"

the presentation in (i). Same for  $G = \{1, -1\} \cong \mathbb{Z}_2$ .

To formalize the notion of a group presentation, we'll need to introduce free groups, and for that we'll need free semigroups.

Def: A **semigroup** is a nonempty set with an associative binary operation (think: "group, with identity; inverses optional")

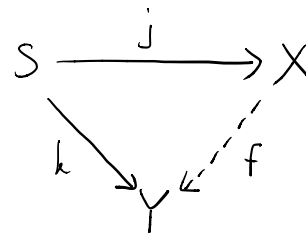
A **semigroup homomorphism** is a function  $f: X \rightarrow Y$  with  $f(x_1 x_2) = f(x_1) f(x_2)$  for all  $x_1, x_2 \in X$ . If  $f$  is bijective, then  $X \cong Y$ .

Def: A **semigroup**  $X$  is **free** on a set  $S$  if there is a function

$j: S \rightarrow X$  s.t. for any other function

$k: S \rightarrow Y$  to a semigroup,  $\exists!$  homom.

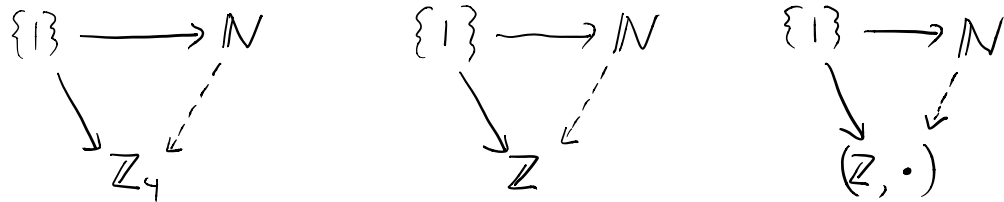
$f: X \rightarrow Y$  s.t.  $fj = k$ .



Example: let  $S = \{1\}$  and  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Then  $(\mathbb{N}, +)$  is a free semigroup on  $S$  with  $j(1) = 1$ . (Verify on HW.)

□

This means that there every semigroup (or group) on one generator is a quotient of  $\mathbb{N}$ , e.g.,



Prop 7.1: If a free semigroup exists on  $S$ , it is unique up to isomorphism.

Pf: HW.

Thm 7.2: If  $S \neq \emptyset$ , then there exists a free semigroup on  $S$ .

Pf: We will construct it explicitly.

Set  $X = S \cup (S \times S) \cup (S \times S \times S) \cup \dots$  "all finite words over  $S$ "

Define a binary operation of concatenation:

$$(a_1, \dots, a_m)(b_1, \dots, b_k) = (a_1, \dots, a_m, b_1, \dots, b_k).$$

This is associative, thus  $X$  is a semigroup.

let  $j: S \rightarrow X$  be the inclusion map:  $j(x) = x$ .

Claim:  $X$  is free on  $S$ .

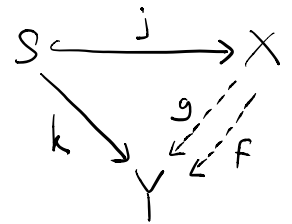
To show this, let  $k: S \rightarrow Y$  be a function to a semigroup.

Define  $f: X \rightarrow Y$  by  $f((a_1, \dots, a_m)) = k(a_1) \dots k(a_m)$ .

Check:  $f$  is a homomorphism:  $fj = k$  ✓

Uniqueness: Say  $g: X \rightarrow Y$  also satisfies  $fj = gj = k$ . □

$$\begin{aligned} \text{Then } g((a_1, \dots, a_m)) &= g((ja_1, \dots, ja_m)) \\ &= g(ja_1) \dots g(ja_m) \\ &= k(a_1) \dots k(a_m) \end{aligned}$$



$$= f(ja_1) \dots f(ja_m) = f((a_1, \dots, a_m)) \Rightarrow f = g. \quad \checkmark$$

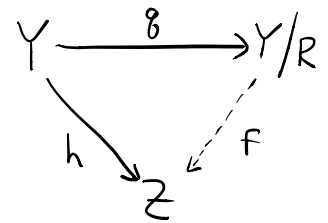
Since  $\exists! f$  s.t.  $fj = k$ ,  $X$  is a free semigroup on  $S$ . □

Prop 7.3: (Quotient semigroups & their universal property)

Let  $Y$  be a semigroup and  $R$  an equivalence relation satisfying  $xRy \wedge zRw \Rightarrow xzRyw$  (i.e., it is well-defined w.r.t. the operation).

Then  $Y/R$  is a semigroup if  $\overline{x} \overline{y} := \overline{xy}$

and it satisfies the following universal property:



"If  $h: Y \rightarrow Z$  is a semigroup homomorphism

with  $fg = h$  and satisfying  $h(x) = h(z)$  whenever  $xRy$ ."

In other words: "every homomorphism respecting  $R$  factors through  $Y/R$ ."

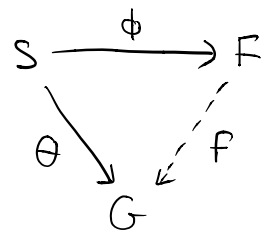
Pf: Existence:  $f([x]) := f(g(x)) = h(x)$ . ✓

Uniqueness: If  $fg = h$ , then  $f = g$  because  $g$  is an epi. □

Def: A group  $F$  is free on a nonempty set  $S$

if  $\exists$  function  $\phi: S \rightarrow F$  s.t. for any other function  $\theta: S \rightarrow G$  to a group,  $\exists!$  homom.

$f: F \rightarrow G$  s.t.  $f\phi = \theta$ .



□ Goal: We'll show that free groups (if they exist) are unique up to isomorphism, and then we'll show they exist by constructing them as quotients of free semigroups.

Prop 7.4: (Uniqueness.) If a free group exists on  $S \neq \emptyset$ , it is unique up to isomorphism, and  $\phi$  is 1-1.

Pf: Uniqueness: Exercise (HW)

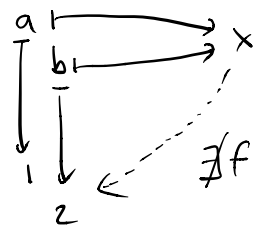
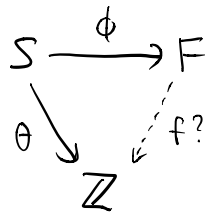
Injectivity of  $\phi$ : Suppose  $\phi$  were not 1-1, and  $a \neq b$  but  $\phi(a) = \phi(b)$ .

Define  $\theta: S \rightarrow \mathbb{Z}$

$$a \mapsto 1$$

$$b \mapsto 2$$

$$c \mapsto 0 \quad c \neq a, b$$



$$\text{Then } 1 = \theta(a) = f\phi(a) = f\phi(b) = \theta(b) = 2.$$

Thus, no such  $f: F \rightarrow \mathbb{Z}$  can exist with  $f\phi = \theta$ , so  $F$  is not free. □

Thm 7.5: (Existence). If  $S \neq \emptyset$ , then there is a free group  $F$  on  $S$ .

Pf: Choose a set  $S'$  with  $|S| = |S'|$ ,  $S \cap S' = \emptyset$ , and put  $T = S \cup S'$ .

(think of  $S'$  as the "inverses" of elts in  $S$ .)

Let  $s \mapsto s'$  be a 1-1 correspondence b/w  $S$  and  $S'$ , and

$$s' \mapsto (s')' = s'' = s \quad \text{the inverse map } S' \rightarrow S.$$

Thus,  $t \mapsto t'$  is a bijection  $T \rightarrow T$ .

Let  $X$  be the free semigroup on  $T$  (exists by Thm. 7.2).

If  $g: X \rightarrow G$  is a homom. to a group, call  $g$  proper if

$$g(s') = g(s)^{-1} \quad \text{for all } s \in S.$$



It follows easily that  $g(t') = g(t)^{-1}$  for all  $t \in T$ . □

★ Define a relation  $R$  on  $X$  where:

$x R y$  iff " $g(x) = g(y)$  holds whenever  $g: X \rightarrow G$  is proper."

Motivation:  $g(x y y^{-1}) = g(x)$  so  $x y y^{-1} R x$ .

Check: •  $R$  is an equiv. relation on  $X$   
•  $x R y$  and  $z R w \Rightarrow x z R y w$ .

Therefore,  $F = X/R$  is a semigroup and  $g: X \rightarrow X/R$  is a homom. (Prop 7.3). Write  $\bar{x} = g(x)$ .

Note that  $\overline{xy} = \bar{x} \bar{y}$  and  $\overline{x^{-1}} = \bar{x}^{-1}$ .

Claim: (i)  $F$  is a group

(ii)  $F$  is free on  $S$ .

Pf of claim:

(i)  $F$  is a group

Choose  $s \in S$ ,  $x \in X$  (under natural inclusion,  $s, s' \in X$ ).

Identity: We'll show  $\overline{ss'} \in X/R$  is the identity elt.

If  $g: X \rightarrow G$  is proper,  $g(ss') = 1$ , and so  $g(ss'x) = g(x)$ .

By definition of  $R$ ,  $x R xss' \iff \bar{x} = \overline{xss'} = \bar{x} \overline{ss'}$

Similarly,  $\overline{ss'} \bar{x} = \bar{x}$ , so  $\overline{ss'} = 1_F$ .

Inverses: let  $x = t_1 t_2 \dots t_k$ ,  $t_i \in T$ .

Then  $(\bar{x})^{-1} = \bar{y}$ , where  $y = t'_k \dots t'_2 t'_1$ .

6

Check: We want to show  $\overline{xy} = 1_F = \overline{ss'}$  for any  $s \in S$ ,  
or equivalently,  $xy \in R_{ss'}$ :

If  $g: X \rightarrow G$  is proper, then

$$\begin{aligned} g(xy) &= g(s_1 s_2 \dots s_k s'_1 \dots s'_l s'_1) \\ &= g(s_1) g(s_2) \dots g(s_k) g(s'_1) \dots g(s'_l) g(s'_1) \\ &= g(s_1) g(s_2) \dots g(s_k) g(s_k)^{-1} \dots g(s_2)^{-1} g(s_1)^{-1} \\ &= 1_G = g(ss') \text{ for any } s \in S. \quad \checkmark \end{aligned}$$

Since  $F = X/R$  is a semigroup with identity  $i$ ; inverses, it is a group.

\* Think of  $F$  as the group of finite words over  $S \cup S^{-1}$  where the binary operation is concatenation.

(ii)  $F$  is free on  $S$ :

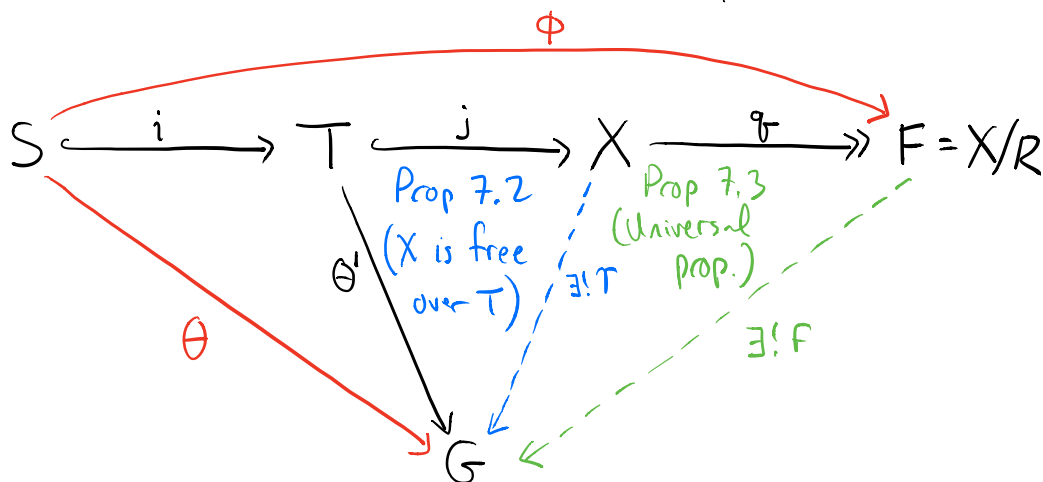
Let  $S \xrightarrow{i} T \hookrightarrow X$  be inclusion maps.

Define  $\phi: S \rightarrow F$  by  $\phi = q \circ j \circ i$ .

Now, let  $\theta: S \rightarrow G$  be any function to a group.

Extend  $\theta$  to  $\theta': T \rightarrow G$  by setting  $\theta'(s') = \theta(s)^{-1} \forall s \in S$ .

\* Goal: Show  $\exists!$  homom.  $f: X \rightarrow G$  s.t.  $f \circ \phi = \theta$ .



7

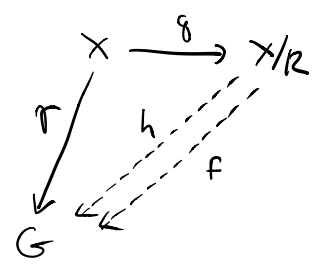
Since  $X$  is free on  $T$ ,  $\exists!$   $\tau: X \rightarrow G$  s.t.  $\tau_j = \theta^i$ . (Prop 7.2)

By Prop 7.3 (Univ. property),  $\exists!$   $f: F \rightarrow G$  s.t.  $f_g = \tau$ .

Note:  $f\phi = f_g j_i = \tau_j i = \theta^i = \theta$ .

Uniqueness of  $f$ : Suppose  $\exists h: F \rightarrow G$  s.t.  $h\phi = \theta$ .

★ If we can show  $h_g = f_g = \tau$ , then uniqueness of  $h$  will follow from Prop 7.3.



Note:  $h_g j_i = \theta = \theta^i = \tau_j i = f_g j_i$ .

Thus  $h_g(s) = f_g(s) (= \tau(s)) \quad \forall s \in S \subseteq X$ .

Need to show  $h_g(s') = f_g(s') (= \tau(s')) \quad \forall s' \in S' \subseteq X$ .

Recall that  $g(s') = g(s)^{-1}$  in  $F = X/R$ .

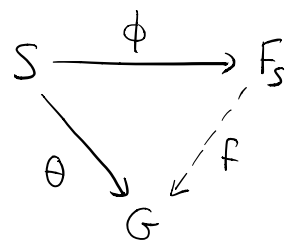
Thus  $h(g(s')) = h(g(s)^{-1}) = (h_g(s))^{-1} = (f_g(s))^{-1} = f(g(s)^{-1}) = f(g(s'))$ . ✓

□

Since  $\phi$  is 1-1, we identify  $s \in S$  with  $\phi(s) \in F$ , and just say  $S \subseteq F$ .

The elements of  $S$  are generators of  $F$ , and we write  $F = F_S = \langle S \rangle$ .

Remark: By definition, any function  $\theta: S \rightarrow G$  (arbitrary  $G$ ) can be extended uniquely to a homom.  $f: F_S \rightarrow G$  s.t.  $f\phi = \theta$ .



Examples:

(1)  $|S|=1$ , say  $S = \{s\}$  and let  $\phi: S \hookrightarrow \mathbb{Z}$   
 $s \mapsto 1$

let  $\theta: S \rightarrow G$  be any map; say  $\theta(s) = g$ .

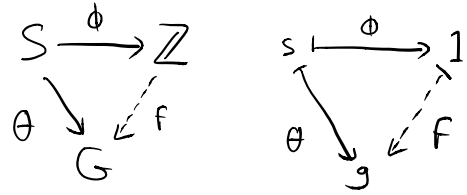
8

Then the homom.  $f: \mathbb{Z} \rightarrow G$   
 $1 \mapsto g$

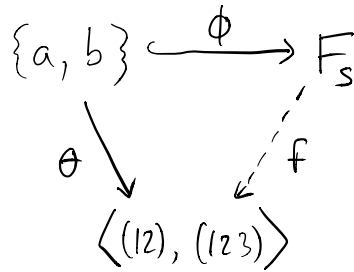
is the unique homom. s.t.  $f\phi = \theta$ .

Thus,  $\mathbb{Z}$  is free on  $S$ .

Note that  $\langle g \rangle \cong \mathbb{Z}$  or  $\mathbb{Z}_n$ , thus every cyclic group is the quotient of the free group on  $\{s\}$  (one generator).



(2) Let  $S = \{a, b\}$ .



Note that  $S_3$  is not cyclic; it has two generators:  $S_3 = \langle (12), (123) \rangle$ .

Let  $\phi: \{a, b\} \hookrightarrow F_S$ .

Define  $\theta: \{a, b\} \rightarrow S_3 = \langle (12), (123) \rangle$   
 $a \mapsto (12)$   
 $b \mapsto (123)$ .

The free group  $F_S$  is the set of all words over  $S = \{a, b\}$  under concatenation, which we write as  $\langle a, b \mid \rangle$ ; i.e., 2 generators, no relations.

The map  $\theta: S \rightarrow S_3$  extends to a unique homom.  $f: F_S \rightarrow S_3$ .

Big idea: The group  $S_3$  is generated by 2 elements, and is a quotient of the free group  $F_S$ , where  $|S| = 2$ .

\* More generally, if  $G = \langle S \rangle$  and  $|S| = n$ , then  $\exists$  homom.  $F_S \twoheadrightarrow G$ , i.e., every group is the quotient of a free group.

Why:  $S \xrightarrow{\phi} F_S$  and  $F$  is surjective because  $\langle S \rangle = G$ . □

Thm 7.6: Suppose  $S, U \neq \emptyset$ . Then  $F_S \cong F_U$  iff  $|S| = |U|$ .

Pf:  $(\Rightarrow)$  Case 1:  $|S| < \infty$

Since  $F_S \cong F_U$ , they have the same number of index-2 subgroups.

Each is characterized uniquely as the kernel of a surjection  $f: F_S \rightarrow \mathbb{Z}_2$ .

Clearly, there are  $2^{|S|} - 1$  such surjections.

So  $F_S$  has  $2^{|S|} - 1$  index-2 subgroups, and  $F_U$  has  $2^{|U|} - 1 \Rightarrow |S| = |U|$ . ✓

Case 2:  $|S| = \infty$ . Set  $T = S \cup S^{-1}$ , so  $|T| = |S|$ .

$$|F_S| \leq 1 + |T| + |T \times T| + |T \times T \times T| + \dots = \sum_0 |T|^n = |S|.$$

Therefore,  $|F_S| = |S|$ , and so  $|S| = |F_S| = |F_U| = |U|$ . ✓

$(\Leftarrow)$  Suppose  $h: S \rightarrow U$  is bijective,

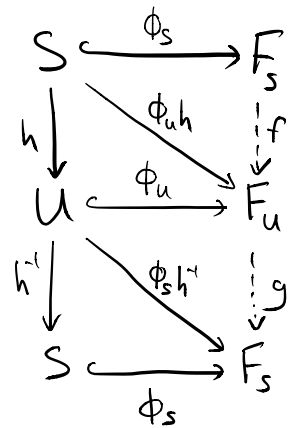
$$\phi_S: S \hookrightarrow F_S, \quad \phi_U: U \hookrightarrow F_U.$$

Then  $\phi_U h: S \hookrightarrow F_U$ , so  $\exists! f: F_S \rightarrow F_U$  s.t.  $f \phi_S = \phi_U h$ .

Similarly,  $\phi_S h^{-1}: U \hookrightarrow F_S$ , so  $\exists! g: F_U \rightarrow F_S$  s.t.  $g \phi_U = \phi_S h^{-1}$ .

Now,  $g f: F_S \rightarrow F_S$  satisfies  $\phi_S = g f \phi_S = 1_S \phi_S$ .

By uniqueness,  $g f = 1_{F_S}$ .



□ Similarly,  $fg = 1_{F_u}$ , so  $f$  &  $g$  are inverse isomorphisms, and  $F_s \cong F_u$ . □

Def: The rank of a free group  $F_S$  is  $|S|$ .

Thm: Subgroups of free groups are free.

Thm: If  $1 < |S|, |U| \leq \aleph_0$ , then  $\exists$  embedding  $F_S \hookrightarrow F_U$ .

Proofs: Involve algebraic topology (covering spaces).

Not surprisingly, the concept of a "free" object can be defined in a categorical setting.

Def: A concrete category is a category  $\mathcal{C}$  where the objects  $A \in \text{Ob}(\mathcal{C})$  have an underlying set structure  $\mathcal{F}(A)$ , and

(i) Every  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  is a function on the underlying sets:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \end{array}$$

(ii) The identity morphism is the identity function on the sets:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow{1_{\mathcal{F}(A)}} & \mathcal{F}(A) \end{array}$$

(iii) Composition of functions agrees with composition of functions on the sets:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) & \xrightarrow{\mathcal{F}(g)} & \mathcal{F}(C) \end{array}$$

Formally, this means that there is a covariant functor  $F$  to  $\square$   
the category  $\text{Set}$ .

Informally: Objects are sets with extra structure (e.g., groups)  
Morphisms are functions with extra structure (e.g., homomorphisms).

Non-examples:

(1) Any directed graph defines a non-concrete category, assuming  
loops & transitive edges.

(2) Let  $G$  be a fixed group.  $\text{Ob}(G) := \{G\}$  (i.e., one object).

$\text{Hom}(G) := \{g : g \in G\}$ . The identity element is  $1_G$ .

Def: Let  $F$  be an object in a concrete category  $\mathcal{C}$ ,  $S \neq \emptyset$ ,  
and  $\phi: S \rightarrow F$  a map of sets. Then  $F$  is free on  $S$   
if for any  $A \in \text{Ob}(\mathcal{C})$  and set map  $\theta: S \rightarrow A$ ,  $\exists! f \in \text{Hom}_{\mathcal{C}}(F, A)$   
s.t.  $f\phi = \theta$  (as maps of sets).

Thm 7.7: Let  $F, F' \in \text{Ob}(\mathcal{C})$  be free objects on  $S, S'$ , respectively,  
where  $|S| = |S'|$ . Then  $F$  and  $F'$  are equivalent.

Pf: HW.

Free objects are universal (i.e., initial) objects in an appropriately  
constructed category, like how co-products were.

12

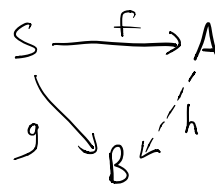
Construction: (Details to be verified on Hw).

Let  $F \in \text{Ob}(\mathcal{C})$  be free on  $S$ , with  $\phi: S \rightarrow F$ .

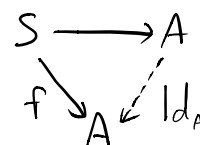
Define a new category  $\mathcal{D}$  as follows:

$\text{Ob}(\mathcal{D})$ : maps of sets  $S \xrightarrow{f} A \in \text{Ob}(\mathcal{C})$

$\text{Hom}(\mathcal{D})$ :  $h \in \text{Hom}_{\mathcal{D}}(S \xrightarrow{f} A, S \xrightarrow{g} B)$  if  $h \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $hf = g$ .

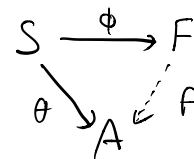


Claim: •  $\text{Id}_A: A \rightarrow A$  is the identity morphism on  $S \xrightarrow{f} A$ .



•  $h \in \text{Hom}_{\mathcal{D}}(S \xrightarrow{f} A, S \xrightarrow{g} B)$  is an equivalence iff  $h \in \text{Hom}_{\mathcal{C}}(A, B)$  is an equivalence.

• If  $F$  is free on  $S$ , then  $S \xrightarrow{\phi} F$  is initial (i.e., universal) in  $\mathcal{D}$ .



Now, back to groups.

Remark: Every group  $G$  generated by  $S$  is a quotient of  $F_S$ .

By the FHT,  $\langle S \rangle = F_S / K$ , for some  $K \trianglelefteq F_S$ .

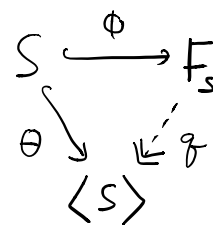
Let  $K = \langle R \rangle$ . Then:

each  $r \in R$  is an element of  $F_S$ ,

i.e., a word in  $S \cup S^{-1}$ .

The quotient  $q: F_S \rightarrow F_S / K$  maps  $r \mapsto rK = K$ ; it "sets  $r = 1$ ."

We say that  $G$  has a set  $S$  of generators  $G$  subject to a set of relations  $\{r = 1 : r \in R\}$ . The elements  $r$  are called relators.





Thus every group  $G = \langle S \rangle$  is isomorphic to a quotient  $F_S/K$ , 13

where  $K := \bigcap_{R \subseteq N_x \trianglelefteq F_S} N_x$ .

We write this as  $G = \langle S | R \rangle := \langle S | r=1 \ \forall r \in R \rangle$ , called a presentation of  $G$ .

Ex:  $C_n = \langle a \mid a^n = 1 \rangle$ .

$\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$ .

$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$ .

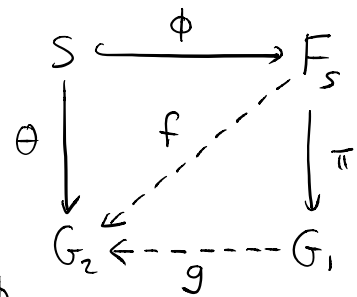
Prop 7.8: Suppose  $G_1 = \langle S | R_1 \rangle$ ,  $G_2 = \langle S | R_2 \rangle$ , and  $R_1 \subseteq R_2$ .

Then  $\exists$  homom.  $g: G_1 \rightarrow G_2$ .

Pf: Assume  $\pi: F_S \rightarrow F_S/K_1 \cong G_1$  is the canonical quotient map.

and  $K_1 = \bigcap_{R \subseteq N_x \trianglelefteq F_S} N_x$ .

Let  $\phi: S \hookrightarrow F_S$  and  $\theta: S \hookrightarrow G_2$  be inclusions.



Since  $F_S$  is free,  $\exists!$   $f: F_S \rightarrow G_2$  s.t.  $\theta = f\phi$ .

Since  $R_1 \subseteq R_2$ ,  $\ker \pi \subseteq \ker f$ , so by the universal prop. of quotient groups,  $\exists!$  homom.  $g: G_1 \rightarrow G_2$  s.t.  $f = g\pi$ . □

To summarize Prop 7.8:

- "Adding relations induces a homomorphism"
- $\langle S | R \rangle$  is the "freest group subject to the relations."

14

Note: Removing generator  $s_i$  is equivalent to adding the relation  $s_i=1$ .

Thus, if  $S_1 \supseteq S_2$  and  $R_1 \subseteq R_2$ , then  $\exists$  homom.  $\langle S_1 | R_1 \rangle \rightarrow \langle S_2 | R_2 \rangle$ .

Actually, we don't need  $S_1 \supseteq S_2$ , but rather just any surjection

$\theta: S_1 \rightarrow S_2$  that "respects relations", i.e.,  $r=1 \Rightarrow \theta(r)=1$ .

More precisely, we have established the following:

Cor 7.9: Suppose  $G_1 = \langle S | R \rangle$  and  $G_2 = \langle S' | R' \rangle$  such that

(i)  $\exists \theta: S \rightarrow S'$ ; say  $\theta(s) = s'$  and extending to  $\theta: R \rightarrow R'$ .

(ii)  $r'=1 \forall r' \in R'$  (i.e.,  $\theta(r)=1 \forall r \in R$ ).

Then  $\exists$  homom.  $g: G_1 \rightarrow G_2$ .

Examples:

(1) Let  $G_1 = \langle a | a^n = 1 \rangle$  and  $G_2 = \langle b | b^m = 1 \rangle$ ,  $m|n$ .

Then  $\theta: a \mapsto b$  and  $\theta(a^n) = b^n = 1$ , so  $\theta$  extends to a homom.

$G_1 \rightarrow G_2$ . (In fact,  $\mathbb{Z}_n \rightarrow \mathbb{Z}_m$ .)

(2) Let  $G = \langle a, b | a^3 = 1, b^2 = 1, abab = 1 \rangle$ .

Note:  $\left. \begin{array}{l} a^3 = 1 \Rightarrow a^{-1} = a^2 \\ b^2 = 1 \Rightarrow b^{-1} = b \end{array} \right\} \Rightarrow abab = 1 \Leftrightarrow ab = ba^{-1} = ba^2$ .

Thus, every element can be written in the form  $b^j a^i$ , where

$i=0,1$  and  $j=0,1,2 \Rightarrow |G| \leq 6$ .

So  $G \cong 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_6$ , or  $S_3$ .

Which one is it?

Consider  $\sigma = (123)$ ,  $\tau = (12)$  in  $S_3 = \langle \sigma, \tau \rangle$ . [15]

Let  $\theta(a) = \sigma$  and  $\theta(b) = \tau$ .

Check:  $\theta(a^3) = \sigma^3 = 1$ ,  $\theta(b^2) = \tau^2 = 1$ ,  $\theta(abab) = \sigma\tau\sigma\tau = 1$ .

By Cor 7.9,  $\exists \theta: G \rightarrow S_3$ , and so  $|G| \geq 6$ .

Therefore,  $G \cong S_3 = \langle a, b \mid a^3, b^2, abab \rangle$ .

(3) Let  $G = \langle x, y \mid xy = y^2x, yx = x^2y \rangle$

Note:  $xy = y^2x \Rightarrow y^{-1}(xy) = yx = x^2y = x(xy) \Rightarrow x = y^{-1}$

Thus,  $1 = xy = y^2x = y(yx) = y \Rightarrow y = 1 \Rightarrow x = 1$ .

Hence,  $G = 1$ .

(4) Let  $G = \langle a, b \mid a^n = b^2 = 1, ab = b^{-1}a \rangle$

Note that  $a^{-1} = a^{n-1}$ ,  $b^{-1} = b$ ,  $ab = ba^{n-1}$ .

Thus, every element can be written as  $a^i b^j$ ,  $0 \leq i < n$ ,  $0 \leq j < 2$ ,

hence  $|G| \leq 2n$ .

To show  $|G| \geq 2n$ , we'll demonstrate  $G \rightarrow D_n$ .

Consider  $D = \langle r, f \rangle$ , where  $r = 2\pi/n$ -rotation,  $f =$  reflection.

Define  $\theta(a) = r$ ,  $\theta(b) = f$ . ✓

Show  $\theta$  "respects relations":  $r^n = f^2 = 1$ ,  $rf = fr^{-1}$

Thus  $\exists \theta: G \rightarrow D_n$ , and since  $|G| \leq |D_n|$ , then  $G \cong D_n$

Fun fact: Given two finitely presented groups  $G_1 = \langle S_1 \mid R_1 \rangle$  and  $G_2 = \langle S_2 \mid R_2 \rangle$  (i.e.,  $|S_i|, |R_i| < \infty$ ), determining whether  $G_1 \cong G_2$  is, in general, computationally undecidable!