

2. Fraction Fields and localizations

□

Throughout, assume R is a commutative ring with 1.

Motivating example: \mathbb{Z} is an integral domain. The field \mathbb{Q} is the "smallest" ring containing it where every element has an inverse.

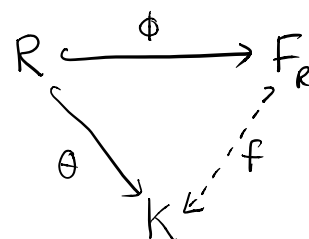
Question: Is there always such a minimal field that has this property?

Def: A **field of fractions** for R is a field

F_R with a monomorphism $\phi: R \hookrightarrow F_R$ s.t. if

$\theta: R \hookrightarrow K$ is a monom. to a field, then

there is a unique homom. $f: F_R \rightarrow K$ s.t. $\theta = f\phi$.



Prop: If $R \neq 0$ has a field of fractions F_R , then it is unique up to isomorphism.

Pf: Exercise (standard uniqueness proof).

Thm: If $R \neq 0$ is an integral domain, then it has a field of fractions.

Pf: Let $X = R \times (R \setminus \{0\})$.

Define an equiv. relation $(a, b) \sim (c, d)$ if $ad = bc$.

[Motivation: $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$.]

Check: Equivalence relation (reflexive, symm, transitive) ✓

Let $F_R = X/\sim$ (set of equiv. classes).

Denote the equiv. class containing (a, b) as a/b .

[2]

Define operations on X/\sim as follows:

$$a/b + c/d = (ad+bc)/bd, \quad a/b \cdot c/d = ac/bd.$$

Check:

- Well-defined. ✓
- Associative. ✓

Thus, F_R is a commutative ring.

$$1 = a/a \text{ for any } a \neq 0.$$

$$0 = 0/b \text{ for any } b \neq 0.$$

$$\text{If } a/b \neq 0, \text{ then } a/b \cdot b/a = 1 \in F_R,$$

thus F_R is a field.

Check universal property:

Define $\phi: R \rightarrow F_R$, $\phi(r) = ra/a$ for any $a \neq 0$.

Easy: ϕ is a monomorphism. ✓

Define $f: F_R \rightarrow K$ by $f(a/b) = \theta(a)\theta(b)^{-1}$.

Check:

- f well-defined. ✓

- f is a monom. ✓

- $f\phi = \theta$ ✓

- uniqueness: Suppose $g: F_R \rightarrow K$ is another monom. s.t.

$$g\phi = f\phi = \theta.$$

$$\begin{aligned} \text{Then } g(r/s) &= g(ra/a \cdot (sa/a)^{-1}) = g(ra/a) g(sa/a)^{-1} = g(\phi(r)) g(\phi(s))^{-1} \\ &= \theta(r)\theta(s)^{-1} = f(r/s). \quad \checkmark \end{aligned}$$

Thus, $g=f$ and F_R is a field of Fractions for R . \square 3

Usually, we identify r with ra/a , and view R as a subring of F_R .

Localizations: Generalizing the construction of F_R .

Let R be comm., and $S \subseteq R$ a semigroup containing no zero divisors.

Goal: Embed R into a ring so that elements of S all have mult. inverses.

Let $X = R \times S$ and define \sim on X by $(a,b) \sim (c,d)$ if $ad=bc$.

The ring $R_S := X/\sim$ is the "smallest" ring containing R s.t. all elements in S have an inverse in R . It is called the localization of R at S .

Ex: Let $R = \mathbb{Z}$, $S = R \setminus (p)$.

This forces all non-multiples of p to have inverses, that is,

$$b^{-1} = 1/b \in R_S \text{ iff } s \nmid b.$$

Thus, all elements of the form a/b , $s \nmid b$ are in R_S .

Exercise: The unique maximal ideal of R_S is $\{pa/b : p \nmid b\}$.

Def: A commutative ring is local if it has a unique maximal ideal.

④

Prop (HW): A ring is local iff its non-units form an ideal.

Prop (HW): If \mathfrak{p} is a prime ideal of a commutative ring, and $S := R \setminus \mathfrak{p}$, then R_S is local.

This is the motivation for the term "localization."