2. Fraction Fields and localizations

Throughout, assume R is a commutative ring with 1.

Motivating example: $\mathbb{Z}$ is an integral domain. The field $\mathbb{Q}$ is the "smallest" ring containing it where every element has an inverse.

Question: Is there always such a minimal field that has this property?

Def: A field of fractions for R is a field F_R with a monomorphism $\phi: R \hookrightarrow F_R$ s.t. if $\theta: R \hookrightarrow K$ is a monom. to a field, then there is a unique homom. $f: F_R \longrightarrow K$ s.t. $\theta = f\phi$.

$$\begin{tikzcd} R \arrow[rr, hook, "\phi"] \arrow[dr, "\theta"'] & & F_R \arrow[dl, dashed, "f"] \\ & K & \end{tikzcd}$$

Prop: If $R \neq 0$ has a field of fractions F_R, then it is unique up to isomorphism.

Pf: Exercise (standard uniqueness proof).

Thm: If $R \neq 0$ is an integral domain, then it has a field of fractions.

Pf: Let $X = R \times (R \setminus \{0\})$.

Define an equiv. relation $(a,b) \sim (c,d)$ if $ad = bc$.

[Motivation: $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$.]

Check: Equivalence relation (reflexive, symm, transitive) ✓

Let $F_R = X/\sim$ (set of equiv. classes).

Denote the equiv. class containing (a,b) as a/b.

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Define operations on $X/\sim$ as follows:

$$a/b + c/d = (ad+bc)/bd, \qquad a/b \cdot c/d = ac/bd.$$

<u>Check</u>: • Well-defined ✓

• Associative. ✓

Thus, F_R is a commutative ring.

$1 = a/a$ for any $a \neq 0$.

$0 = 0/b$ for any $b \neq 0$.

If $a/b \neq 0$, then $a/b \cdot b/a = 1 \in F_R$,

thus F_R is a field.

<u>Check universal property</u>:

Define $\phi : R \hookrightarrow F_R$, $\phi(r) = ra/a$ for any $a \neq 0$.

<u>Easy</u>: ϕ is a monomorphism. ✓

Define $f : F_R \longrightarrow K$ by $f(a/b) = \theta(a)\,\theta(b)^{-1}$.

<u>Check</u>: • f well-defined ✓

• f is a monom. ✓

• $f\phi = \theta$ ✓

• uniqueness: Suppose $g : F_R \longrightarrow K$ is another monom. s.t. $g\phi = f\phi = \theta$.

Then $g(r/s) = g\left(ra/a \cdot (sa/a)^{-1}\right) = g(ra/a)\, g(sa/a)^{-1} = g(\phi(r))\, g(\phi(s))^{-1}$

$= \theta(r)\,\theta(s) = f(r/s)$. ✓

Thus, $g = f$ and F_R is a field of fractions for R. $\square$

Usually, we identify r with ra/a, and view R as a subring of F_R.

Localizations: Generalizing the construction of F_R.

Let R be comm., and $S \subseteq R$ a semigroup containing no zero divisors.

Goal: Embed R into a ring so that elements of S all have mult. inverses.

Let $X = R \times S$ and define $\sim$ on X by $(a,b) \sim (c,d)$ if $ad = bc$.

The ring $R_S := X/\sim$ is the "smallest" ring containing R s.t. all elements in S have an inverse in R. It is called the localization of R at S.

Ex: Let $R = \mathbb{Z}$, $S = R \setminus (p)$.

This forces all non-multiples of p to have inverses, that is,

$b^{-1} = 1/b \in R_S$ iff $5 \nmid b$.

Thus, all elements of the form a/b, $5 \nmid b$ are in R_S.

Exercise: The unique maximal ideal of R_S is $\{pa/b : p \nmid b\}$.

Def: A commutative ring is local if it has a unique maximal ideal.

<u>Prop</u> (HW): A ring is local iff its non-units form an ideal.

<u>Prop</u> (HW): If P is a prime ideal of a commutative ring, and $S := R \setminus P$, then R_S is local.

This is the motivation for the term "localization."