Section 2.6: Divisibility and factorization

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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Motivation

A ring is in some sense, a generalization of the familiar number systems like $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, where we are allowed to add, subtract, and multiply.

Two key properties about these structures are:

- multiplication is commutative,
- there are no (nonzero) zero divisors.

Blanket assumption

Throughout this lecture, unless explicitly mentioned otherwise, R is assumed to be an integral domain, and we will define $R^* := R \setminus \{0\}$.

The integers have several basic properties that we usually take for granted:

- every nonzero number can be factored uniquely into primes;
- any two numbers have a unique greatest common divisor and least common multiple;
- there is a Euclidean algorithm, which can find the gcd of two numbers.

Surprisingly, these need not always hold in integrals domains. We would like to understand this better.

Divisibility

Definition

If $a, b \in R$, say that a divides b, or b is a multiple of a if $b = ac$ for some $c \in R$. We write $a \mid b$.

If $a \mid b$ and $b \mid a$, then a and b are associates, written $a \sim b$.

Examples

- In $\mathbb{Z}$: n and $-n$ are associates.
- In $\mathbb{R}[x]$: $f(x)$ and $c \cdot f(x)$ are associates for any $c \neq 0$.
- The only associate of 0 is itself.
- The associates of 1 are the units of R.

Exercise

Two elements $a, b \in R$ are associates if and only if $a = bu$ for some unit $u \in U(R)$.

This defines an equivalence relation on R, and partitions R into equivalence classes.

Irreducibles and primes

Note that units divide everything: if $b \in R$ and $u \in U(R)$, then $u \mid b$.

Definition

If $b \in R$ is not a unit, and the only divisors of b are units and associates of b, then b is irreducible.

An element $p \in R$ is prime if p is not a unit, and $p \mid ab$ implies $p \mid a$ or $p \mid b$.

Proposition

If $0 \neq p \in R$ is prime, then p is irreducible.

Proof

Suppose p is prime but not irreducible. Then $p = ab$ with $a, b \notin U(R)$.

Then (wlog) $p \mid a$, so $a = pc$ for some $c \in R$. Now,

$$p = ab = (pc)b = p(cb)\,.$$

This means that $cb = 1$, and thus $b \in U(R)$, a contradiction. $\square$

Irreducibles and primes

Caveat: Irreducible $\not\Rightarrow$ prime

Consider the ring $R_{-5} := \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$.

$$3 \mid (2+\sqrt{-5})(2-\sqrt{-5}) = 9 = 3 \cdot 3\,,$$

but $3 \nmid 2+\sqrt{-5}$ and $3 \nmid 2-\sqrt{-5}$.

Thus, 3 is irreducible in R_{-5} but *not* prime.

When irreducibles fail to be prime, we can lose nice properties like unique factorization.

Things can get really bad: not even the *lengths* of factorizations into irreducibles need be the same!

For example, consider the ring $R = \mathbb{Z}[x^2, x^3]$. Then

$$x^6 = x^2 \cdot x^2 \cdot x^2 = x^3 \cdot x^3.$$

The element $x^2 \in R$ is not prime because $x^2 \mid x^3 \cdot x^3$ yet $x^2 \nmid x^3$ in R (note: $x \notin R$).

Principal ideal domains

Fortunately, there is a type of ring where such "bad things" don't happen.

Definition

An ideal I generated by a single element $a \in R$ is called a principal ideal. We denote this by $I = (a)$.

If every ideal of R is principal, then R is a principal ideal domain (PID).

Examples

The following are all PIDs (stated without proof):

- The ring of integers, $\mathbb{Z}$.
- Any field F.
- The polynomial ring $F[x]$ over a field.

As we will see shortly, PIDs are "nice" rings. Here are some properties they enjoy:

- pairs of elements have a "greatest common divisor" & "least common multiple";
- irreducible $\Rightarrow$ prime;
- Every element factors uniquely into primes.

Greatest common divisors & least common multiples

Proposition

If $I \subseteq \mathbb{Z}$ is an ideal, and $a \in I$ is its smallest positive element, then $I = (a)$.

Proof

Pick any positive $b \in I$. Write $b = aq + r$, for $q, r \in \mathbb{Z}$ and $0 \leq r < a$.

Then $r = b - aq \in I$, so $r = 0$. Therefore, $b = qa \in (a)$. $\square$

Definition

A common divisor of $a, b \in R$ is an element $d \in R$ such that $d \mid a$ and $d \mid b$.

Moreover, d is a greatest common divisor (GCD) if $c \mid d$ for all other common divisors c of a and b.

A common multiple of $a, b \in R$ is an element $m \in R$ such that $a \mid m$ and $b \mid m$.

Moreover, m is a least common multiple (LCM) if $m \mid n$ for all other common multiples n of a and b.

Nice properties of PIDs

Proposition

If R is a PID, then any $a, b \in R^*$ have a GCD, $d = \gcd(a, b)$.

It is *unique up to associates*, and can be written as $d = xa + yb$ for some $x, y \in R$.

Proof

Existence. The ideal generated by a and b is

$$I = (a, b) = \{ua + vb : u, v \in R\} .$$

Since R is a PID, we can write $I = (d)$ for some $d \in I$, and so $d = xa + yb$.

Since $a, b \in (d)$, both $d \mid a$ and $d \mid b$ hold.

If c is a divisor of a & b, then $c \mid xa + yb = d$, so d is a GCD for a and b. ✓

Uniqueness. If d' is another GCD, then $d \mid d'$ and $d' \mid d$, so $d \sim d'$. ✓ □

Nice properties of PIDs

Corollary

If R is a PID, then every irreducible element is prime.

Proof

Let $p \in R$ be irreducible and suppose $p \mid ab$ for some $a, b \in R$.

If $p \nmid a$, then $\gcd(p, a) = 1$, so we may write $1 = xa + yp$ for some $x, y \in R$. Thus

$$b = (xa + yp)b = x(ab) + (yb)p\,.$$

Since $p \mid x(ab)$ and $p \mid (yb)p$, then $p \mid x(ab) + (yb)p = b$. $\square$

Not surprisingly, least common multiples also have a nice characterization in PIDs.

Proposition (HW)

If R is a PID, then any $a, b \in R^*$ have an LCM, $m = \text{lcm}(a, b)$.

It is *unique up to associates*, and can be characterized as a generator of the ideal $I := (a) \cap (b)$.

Unique factorization domains

Definition

An integral domain is a unique factorization domain (UFD) if:

(i) Every nonzero element is a product of irreducible elements;

(ii) Every irreducible element is prime.

Examples

1. $\mathbb{Z}$ is a UFD: Every integer $n \in \mathbb{Z}$ can be uniquely factored as a product of irreducibles (primes):
$$n = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}.$$
This is the *fundamental theorem of arithmetic*.
2. The ring $\mathbb{Z}[x]$ is a UFD, because every polynomial can be factored into irreducibles. But it is not a PID because the following ideal is not principal:
$$(2, x) = \{f(x) : \text{ the constant term is even}\}.$$
3. The ring R_{-5} is not a UFD because $9 = 3 \cdot 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$.
4. We've shown that (ii) holds for PIDs. Next, we will see that (i) holds as well.

Unique factorization domains

Theorem

If R is a PID, then R is a UFD.

Proof

We need to show Condition (i) holds: every element is a product of irreducibles. A ring is Noetherian if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

stabilizes, meaning that $I_k = I_{k+1} = I_{k+2} = \cdots$ holds for some k.

Suppose R is a PID. It is not hard to show that R is Noetherian (HW). Define

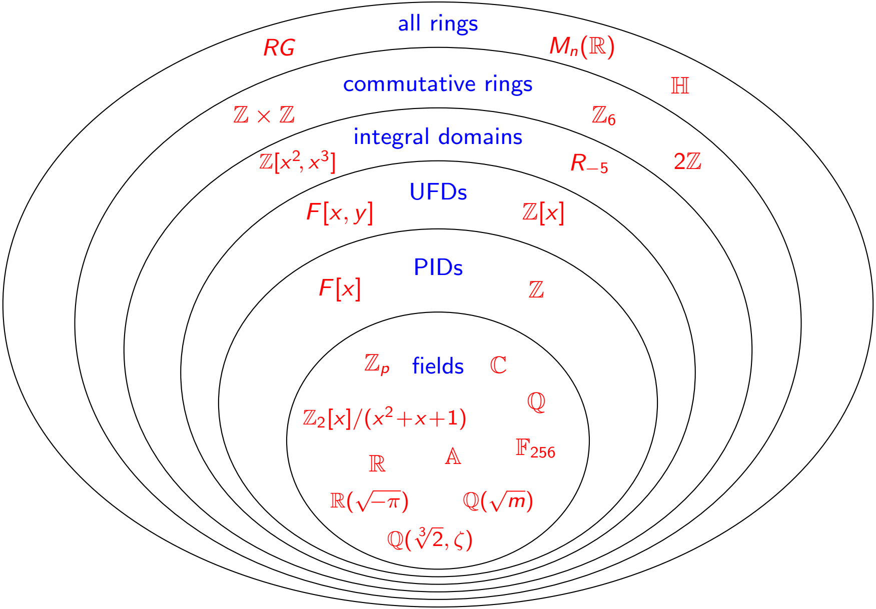
$$X = \{a \in R^* \setminus U(R) : a \text{ can't be written as a product of irreducibles}\}.$$

If $X \neq \emptyset$, then pick $a_1 \in X$. Factor this as $a_1 = a_2 b$, where $a_2 \in X$ and $b \notin U(R)$. Then $(a_1) \subsetneq (a_2) \subsetneq R$, and repeat this process. We get an ascending chain

$$(a_1) \subsetneq (a_2) \subsetneq (a_3) \subsetneq \cdots$$

that does not stabilize. This is impossible in a PID, so $X = \emptyset$. □

Summary of ring types



The Euclidean algorithm

Around 300 B.C., Euclid wrote his famous book, the *Elements*, in which he described what is now known as the Euclidean algorithm:

Proposition VII.2 (Euclid's *Elements*)

Given two numbers not prime to one another, to find their greatest common measure.

The algorithm works due to two key observations:

- If $a \mid b$, then $\gcd(a,b) = a$;
- If $a = bq + r$, then $\gcd(a,b) = \gcd(b,r)$.

This is best seen by an example: Let $a = 654$ and $b = 360$.

$$
\begin{aligned}
654 &= 360 \cdot 1 + 294 \qquad & \gcd(654,360) &= \gcd(360,294) \\
360 &= 294 \cdot 1 + 66 & \gcd(360,294) &= \gcd(294,66) \\
294 &= 66 \cdot 4 + 30 & \gcd(294,66) &= \gcd(66,30) \\
66 &= 30 \cdot 2 + 6 & \gcd(66,30) &= \gcd(30,6) \\
30 &= 6 \cdot 5 & \gcd(30,6) &= 6.
\end{aligned}
$$

We conclude that $\gcd(654,360) = 6$.

Euclidean domains

Loosely speaking, a Euclidean domain is any ring for which the Euclidean algorithm still works.

Definition

An integral domain R is Euclidean if it has a degree function $d\colon R^* \to \mathbb{Z}$ satisfying:

(i) non-negativity: $d(r) \geq 0 \quad \forall r \in R^*$.

(ii) monotonicity: $d(a) \leq d(ab)$ for all $a, b \in R^*$.

(iii) division-with-remainder property: For all $a, b \in R$, $b \neq 0$, there are $q, r \in R$ such that

$$a = bq + r \qquad \text{with} \qquad r = 0 \quad \text{or} \quad d(r) < d(b)\,.$$

Note that Property (ii) could be restated to say: *If $a \mid b$, then $d(a) \leq d(b)$*;

Examples

- $R = \mathbb{Z}$ is Euclidean. Define $d(r) = |r|$.
- $R = F[x]$ is Euclidean if F is a field. Define $d(f(x)) = \deg f(x)$.
- The Gaussian integers $R_{-1} = \mathbb{Z}[\sqrt{-1}] = \{a + bi : a, b \in \mathbb{Z}\}$ is Euclidean with degree function $d(a + bi) = a^2 + b^2$.

Euclidean domains

Proposition

If R is Euclidean, then $U(R) = \{x \in R^* : d(x) = d(1)\}$.

Proof

$\subseteq$": First, we'll show that associates have the same degree. Take $a \sim b$ in R^*:

$$\begin{array}{lcl} a \mid b & \Longrightarrow & d(a) \leq d(b) \\ b \mid a & \Longrightarrow & d(b) \leq d(a) \end{array} \qquad \Longrightarrow \qquad d(a) = d(b).$$

If $u \in U(R)$, then $u \sim 1$, and so $d(u) = d(1)$. ✓

"$\supseteq$": Suppose $x \in R^*$ and $d(x) = d(1)$.

Then $1 = qx + r$ for some $q \in R$ with either $r = 0$ or $d(r) < d(x) = d(1)$.

If $r \neq 0$, then $d(1) \leq d(r)$ since $1 \mid r$.

Thus, $r = 0$, and so $qx = 1$, hence $x \in U(R)$. ✓ $\square$

Euclidean domains

Proposition

If R is Euclidean, then R is a PID.

Proof

Let $I \neq 0$ be an ideal and pick some $b \in I$ with $d(b)$ minimal.

Pick $a \in I$, and write $a = bq + r$ with either $r = 0$, or $d(r) < d(b)$.

This latter case is impossible: $r = a - bq \in I$, and by minimality, $d(b) \leq d(r)$.

Therefore, $r = 0$, which means $a = bq \in (b)$. Since a was arbitrary, $I = (b)$. $\square$

Exercises.

(i) The ideal $I = (3, 2 + \sqrt{-5})$ is not principal in R_{-5}.

(ii) If R is an integral domain, then $I = (x, y)$ is not principal in $R[x, y]$.

Corollary

The rings R_{-5} (not a PID or UFD) and $R[x, y]$ (not a PID) are not Euclidean.

Algebraic integers

The algebraic integers are the roots of *monic* polynomials in $\mathbb{Z}[x]$. This is a subring of the algebraic numbers (roots of all polynomials in $\mathbb{Z}[x]$).

Assume $m \in \mathbb{Z}$ is square-free with $m \neq 0, 1$. Recall the quadratic field

$$\mathbb{Q}(\sqrt{m}) = \left\{ p + q\sqrt{m} \mid p, q \in \mathbb{Q} \right\}.$$

Definition

The ring R_m is the set of algebraic integers in $\mathbb{Q}(\sqrt{m})$, i.e., the subring consisting of those numbers that are roots of monic quadratic polynomials $x^2 + cx + d \in \mathbb{Z}[x]$.

Facts

- R_m is an integral domain with 1.
- Since m is square-free, $m \not\equiv 0 \pmod 4$. For the other three cases:

$$R_m = \begin{cases} \mathbb{Z}[\sqrt{m}] = \left\{ a + b\sqrt{m} : a, b \in \mathbb{Z} \right\} & m \equiv 2 \text{ or } 3 \pmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{m}}{2}\right] = \left\{ a + b\left(\frac{1+\sqrt{m}}{2}\right) : a, b \in \mathbb{Z} \right\} & m \equiv 1 \pmod 4 \end{cases}$$

- R_{-1} is the Gaussian integers, which is a PID. (easy)
- R_{-19} is a PID. (hard)

Algebraic integers

Definition

For $x = r + s\sqrt{m} \in \mathbb{Q}(\sqrt{m})$, define the norm of x to be

$$N(x) = (r + s\sqrt{m})(r - s\sqrt{m}) = r^2 - ms^2 .$$

R_m is norm-Euclidean if it is a Euclidean domain with $d(x) = |N(x)|$.

Note that the norm is multiplicative: $N(xy) = N(x)N(y)$.

Exercises

Assume $m \in \mathbb{Z}$ is square-free, with $m \neq 0, 1$.

- $u \in U(R_m)$ iff $|N(u)| = 1$.
- If $m \geq 2$, then $U(R_m)$ is infinite.
- $U(R_{-1}) = \{\pm 1, \pm i\}$ and $U(R_{-3}) = \{\pm 1, \pm \frac{1 \pm \sqrt{-3}}{2}\}$.
- If $m = -2$ or $m < -3$, then $U(R_m) = \{\pm 1\}$.

Euclidean domains and algebraic integers

Theorem

R_m is norm-Euclidean iff

$$m \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$$

Theorem (D.A. Clark, 1994)

The ring R_{69} is a Euclidean domain that is *not* norm-Euclidean.

Let $\alpha = (1 + \sqrt{69})/2$ and $c > 25$ be an integer. Then the following degree function works for R_{69}, defined on the prime elements:

$$d(p) = \begin{cases} |N(p)| & \text{if } p \neq 10 + 3\alpha \\ c & \text{if } p = 10 + 3\alpha \end{cases}$$

Theorem

If $m < 0$ and $m \notin \{-11, -7, -3, -2, -1\}$, then R_m is not Euclidean.

Open problem

Classify which R_m's are PIDs, and which are Euclidean.

PIDs that are not Euclidean

Theorem

If $m < 0$, then R_m is a PID iff

$$m \in \{\underbrace{-1, -2, -3, -7, -11}_{\text{Euclidean}}, -19, -43, -67, -163\}.$$

Recall that R_m is norm-Euclidean iff

$$m \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$$

Corollary

If $m < 0$, then R_m is a PID that is not Euclidean iff $m \in \{-19, -43, -67, -163\}$.

Algebraic integers

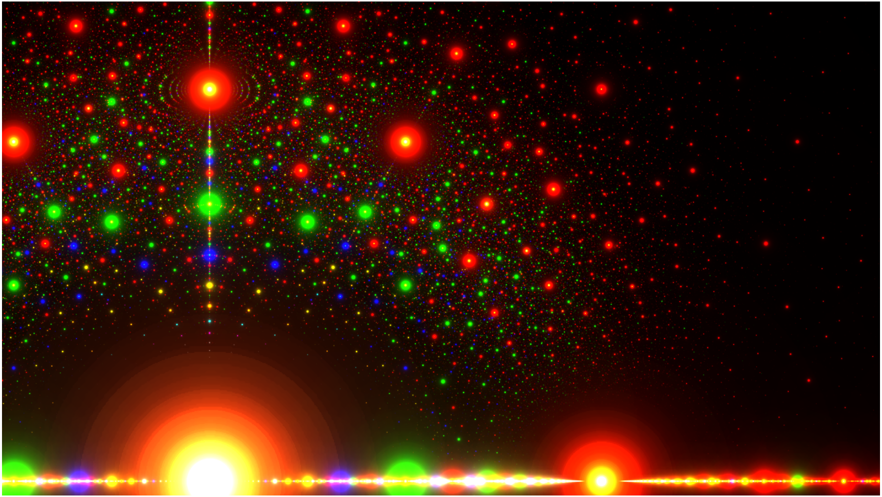


Figure: Algebraic numbers in the complex plane. Colors indicate the coefficient of the leading term: red $= 1$ (algebraic integer), green $= 2$, blue $= 3$, yellow $= 4$. Large dots mean fewer terms and smaller coefficients. Image from Wikipedia (made by Stephen J. Brooks).

Algebraic integers

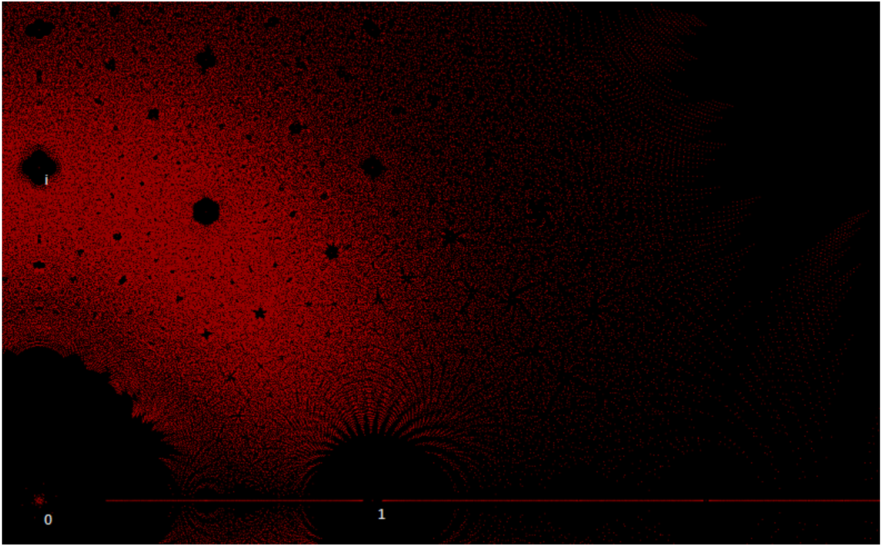


Figure: Algebraic integers in the complex plane. Each red dot is the root of a monic polynomial of degree ≤ 7 with coefficients from $\{0, \pm1, \pm2, \pm3, \pm4, \pm5\}$. From Wikipedia.

Summary of ring types