$\boxed{\text{Infinity}}$ Tues 8/27

What do we mean by infinity?

Numbers? Lines? Space? Something else?

How does infinity arise in art & architecture?

Can we do math with infinity?

$\frac{1}{0} = \infty$, $\frac{-1}{0} = -\infty$, $\frac{1}{\infty} = 0$, $\frac{0}{0} = ?$ $\infty + \infty = \infty$, $\infty - \infty = ?$ $\frac{\infty}{\infty} = ?$

<u>Bird example:</u> 2 farmers plant 1 seed/day.

A bird eats one seed every 4 days.

day 4 day 8

Farmer 1: 1 2 3 $\cancel{4}$ 5 6 7 $\cancel{8}$ 9 10 ...

Farmer 2: $\cancel{1}$ $\cancel{2}$ 3 4 5 6 7 8 9 10 ...

day 4 day 8

How many seeds are left "at the end of time"?

Wed 8/28

<u>Question:</u> Are all infinities the same "size"?

What does this even mean?

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$ "Natural numbers"

$2\mathbb{N} = \{2, 4, 6, 8, 10, 12, \ldots\}$

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ "integers"

$= \{0, 1, -1, 2, -2, \ldots\}$

$\mathbb{Q} = \{\frac{a}{b} : a \in \mathbb{Z},\ b \in \mathbb{N},\ \gcd(a,b) = 1\}$ "rational numbers"

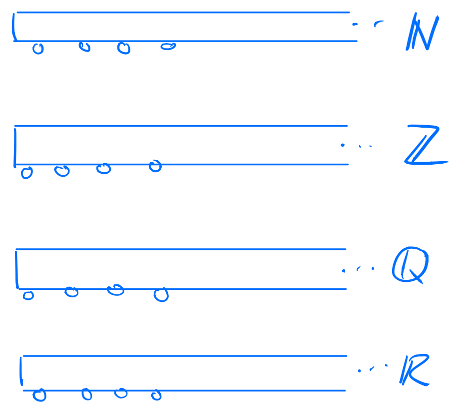
$\mathbb{R} = \{\text{all real numbers}\}$ "real numbers"

Note: $2\mathbb{N} \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$.

Hilbert's hotel:

1	2	3	4	5	6	7	8	...

Infinite buses arrive:



How to give them rooms?

Proof that $|\mathbb{N}| = |\mathbb{Q}^+|$

⋮	⋮	⋮	⋮	⋮	⋮	
11 $5/1$	$5/2$ 12	$5/3$	$5/4$	$5/5$	$5/6$	...
6 $4/1$	$4/2$	$4/3$ 13	$4/4$	$4/5$	$4/6$	...
$3/1$ 5	$3/2$ 7	$3/3$	$3/4$ 14	$3/5$	$3/6$	...
2 $2/1$	$2/2$	$2/3$ 8	$2/4$	$2/5$ 15	$2/6$	...
1 $1/1$	$1/2$ 3	$1/3$ 4	$1/4$ 9	$1/5$ 10	$1/6$ 16	...

Proof that $|\mathbb{N}| < |\mathbb{R}|$. "Cantor's diagonal argument," 1891

Suppose it were false, and we could order them:

Room 1	$0.\boxed{1}234802009\ldots$
Room 2	$3.8\boxed{8}9201348 1\ldots$
Room 3	$16.00\boxed{2}1986623\ldots$
Room 4	$4.088\boxed{1}254565\ldots$
Room 5	$5.1823\boxed{9}39514\ldots$
Room 6	$3.14159\boxed{2}8535\ldots$

The following number is <u>not</u> on this list:

$0.293203\ldots$ (add 1 to each boxed digit).

This contradicts the hypothesis that we could have ordered the real #'s in the first place.

$\boxed{\text{Timeline}}$ <u>1889</u>: Cantor proved $|\mathbb{Q}| < |\mathbb{R}|$

Cantor then worked on "continuum hypothesis": is there an infinity between these two?

<u>1899–1917</u>: Cantor in & out of sanatorium.

<u>1940</u>: Gödel's incompleteness theorem

<u>1963</u>: Cohen proves the continuum hypothesis is undecidable.

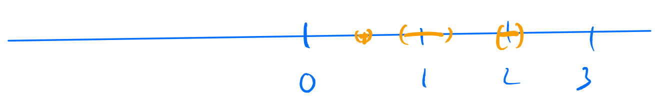
<u>Goldbach conjecture</u>: Every even number >2 can be written as a sum of two primes.

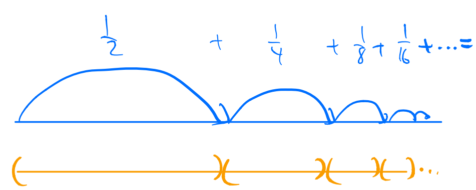
Still unproven. Is it also undecidable???

Fri 8/30

Two Fun Facts about $|\mathbb{N}| = |\mathbb{Q}| < |\mathbb{R}|$:

① We can "cover" the rationals with intervals of total length 1.





Put n^{th} "ball" $(\longleftrightarrow)$, of length $\left(\frac{1}{2}\right)^n$, around the n^{th} rational number.

② There are countably many computer programs (strings of 0, 1), but uncountably many real numbers. So most are uncomputable!

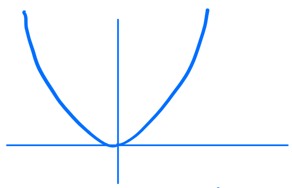
Review of Functions

Domain: All possible inputs

Range: All possible outputs

Examples:

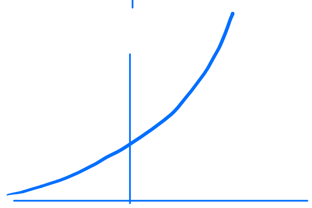
① $f(x) = x^2$



D: $(-\infty, \infty)$ or $\mathbb{R}$ or $-\infty < x < \infty$

R: $[0, \infty)$ or $f(x) \geq 0$

② $f(x) = e^x$



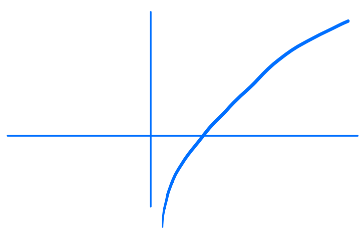
D: $(-\infty, \infty)$

R: $(0, \infty)$ or $f(x) > 0$

③ $f(x) = \ln x$

Recall: $e^{\ln x} = \ln e^x = x$

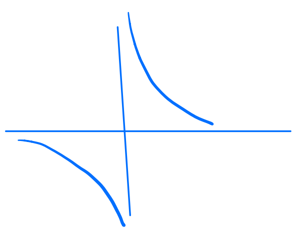
"inverse functions"



D: $(0, \infty)$

R: $(-\infty, \infty)$

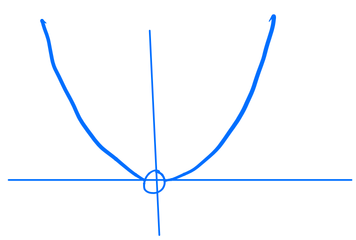
④ $f(x) = \frac{1}{x}$



D: $x \neq 0$ or $(-\infty, 0) \cup (0, \infty)$

R: $f(x) \neq 0$ " "

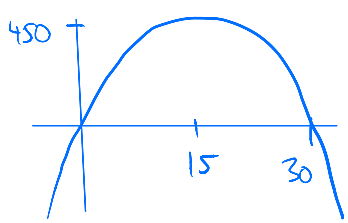
⑤ $f(x) = \frac{x^3}{x}$ $\left(= x^2 \text{ if } x \neq 0\right)$



D: $x \neq 0$

R: $f(x) > 0$

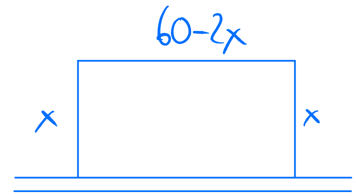
⑥ $f(x) = x(60 - 2x)$



D: $(-\infty, \infty)$

R: $f(x) \leq 450$

(7) Let $A(x)$ = area of a rectangular pen built with 60 feet of fence, with one side an existing brick wall $\longrightarrow$



D: $0 < x < 30$

R: $0 < A(x) < 450$

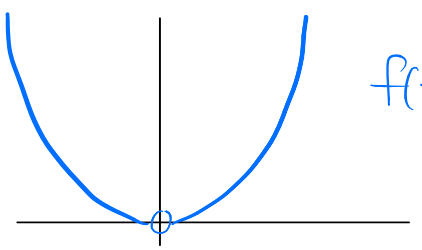
Note: $A(x) = x(60-2x)$

Mon 9/2

Limits

Def (informal): $\lim_{x \to a} f(x)$ is what $f(a)$ "should be"

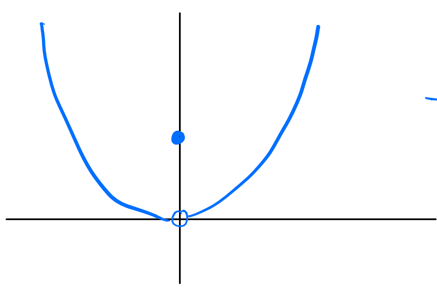
Ex 1:



$$f(x) = \begin{cases} x^2 & x \neq 0 \\ ? & x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = 0$$

Ex 2:



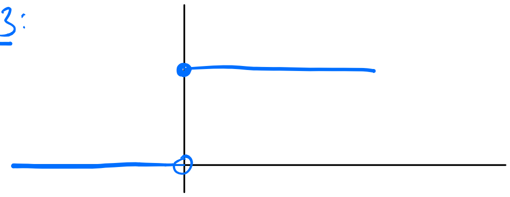
$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 2 & x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = 0.$$

We can define the:

- left-hand limit $\lim_{x \to a^-} f(x)$
- right-hand limit $\lim_{x \to a^+} f(x)$

Ex 3:



$\lim_{x\to 0^-} f(x) = 0$

$\lim_{x\to 0^+} f(x) = 0$

So $\lim_{x\to 0} F(x)$ DNE "does not exist"

Note that $f(0)=0$.

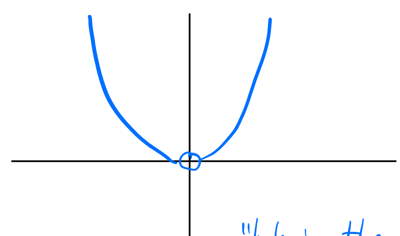
Def: The limit of $f(x)$ at $x=a$ exists when the left-hand & right-hand limits of $f(x)$ at $x=a$ exist. We write

$$\lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

Def: The function $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$.

More examples

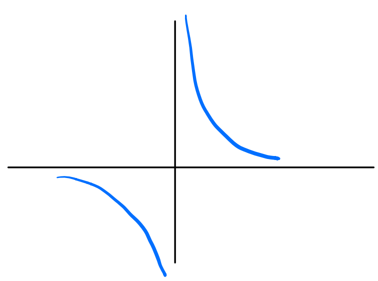
Ex 4: $f(x) = \dfrac{x^3}{x} = \begin{cases} x^2 & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}$



"hole in the graph"

Limits at infinity: We can also define $\lim_{x\to\infty} f(x)$, though it may or may not exist.

Ex 5: $f(x) = \dfrac{1}{x}$



$\lim_{x\to 0^-} f(x) = -\infty$ (DNE)

$\lim_{x\to 0^+} f(x) = \infty$ (DNE)

$\lim_{x\to 0} f(x)$ DNE

$\lim_{x\to\infty} f(x) = 0$

$\lim_{x\to -\infty} f(x) = 0$

Sequences can have limits.

Ex 6: Let $a_n = \frac{1}{n}$ (defines the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$)

We say that $\lim_{n\to\infty} a_n = 0$.

Many sequences don't have limits, eg,

$1, 2, 3, 4, 5, 6, \ldots$

$0, 1, 0, 1, 0, 1, \ldots$

$1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots$

How to find $\lim_{x\to a} f(x)$

- First, try plugging in $x = a$.
- If that doesn't work, graph $f(x)$, try to determine $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$

☆ Limits generally behave like you'd expect

- $\lim_{x\to a}\left[f(x)+g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$, provided these exist.

- $\lim_{x\to a} f(x)\, g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

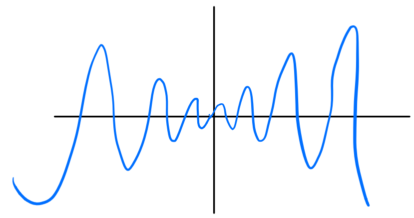
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Ex 7: $\lim_{x\to 0} \frac{2x^2+3x}{x} = \lim_{x\to 0} \frac{2x^2}{x} + \lim_{x\to 0} \frac{3x}{x} = \lim_{x\to 0} 2x + \lim_{x\to 0} 3$

$= 0 + 3 = 3$

Ex 8: $\lim_{x\to 0} x \sin x = \lim_{x\to 0} x \cdot \lim_{x \to 0} \sin x$

$= 0 \cdot 0 = 0$



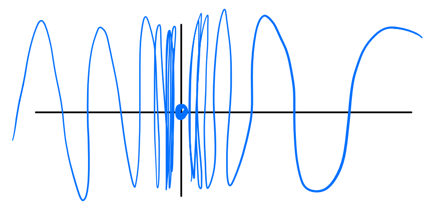
what could go wrong:

Ex 9: $\lim_{x\to 0} \frac{\sin x}{x} \stackrel{?}{=} \lim_{x\to 0} \frac{1}{x} \cdot \lim_{x\to 0} \sin x$

$\text{"="} \quad \infty \cdot 0 = ???$

Other things can get weird

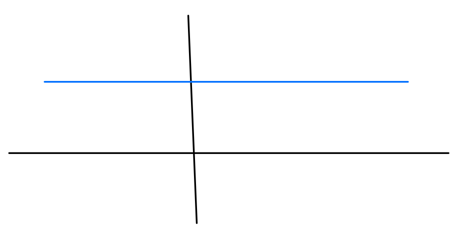
Just for fun... let $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



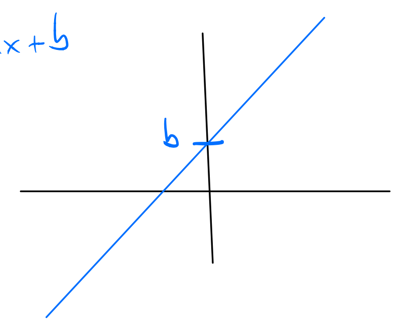
Wed 9/3

Let's review some basic functions & how to graph them.

Constants: $f(x) = 1$



Linear: $f(x) = mx + b$



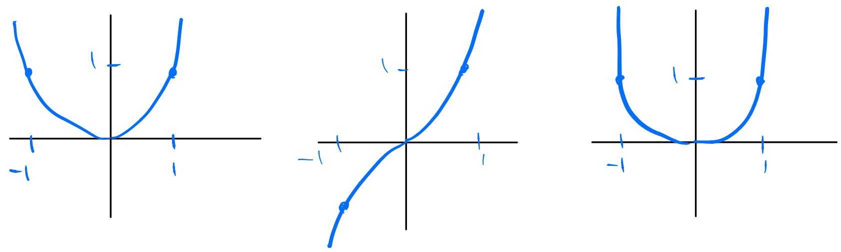
b = y-intercept

m = slope

↳ "$\frac{\text{rise}}{\text{run}}$"

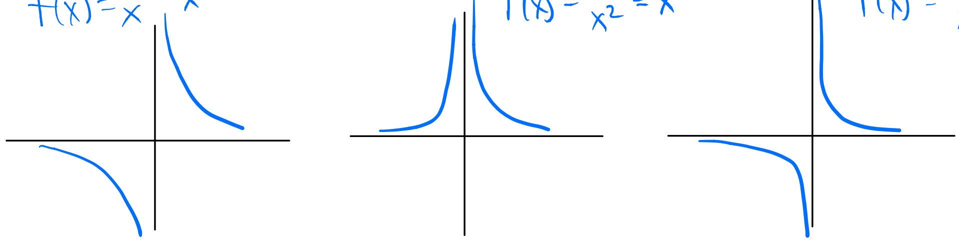
Polynomials

$f(x) = x^2$ $\quad$ $f(x) = x^3$ $\quad$ $f(x) = x^4$ $\quad \ldots$



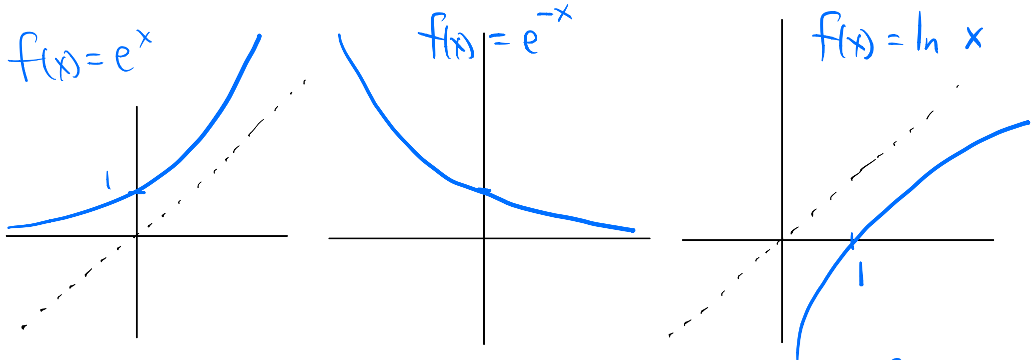
x^{-n}

$f(x) = \frac{1}{x} = x^{-1}$ $\quad$ $f(x) = \frac{1}{x^2} = x^{-2}$ $\quad$ $f(x) = \frac{1}{x^3} = x^{-3} \ldots$



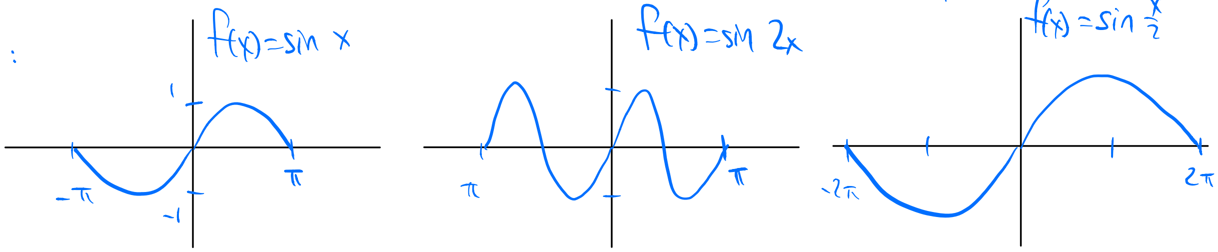
Exponentials:

$f(x) = e^x$ $\quad$ $f(x) = e^{-x}$ $\quad$ $f(x) = \ln x$

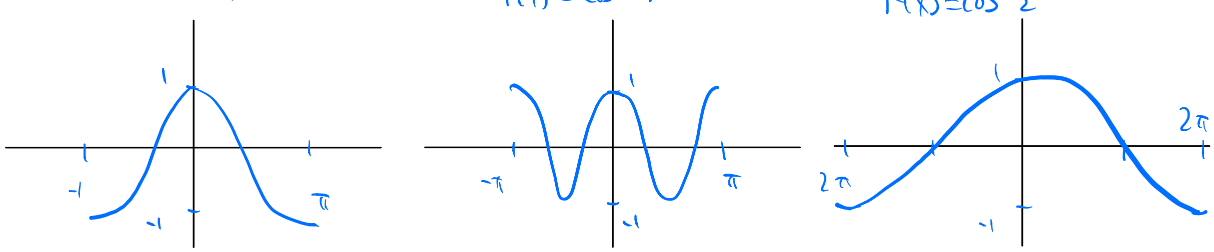


Trig:

$f(x) = \sin x$ $\quad$ $f(x) = \sin 2x$ $\quad$ $f(x) = \sin \frac{x}{2}$

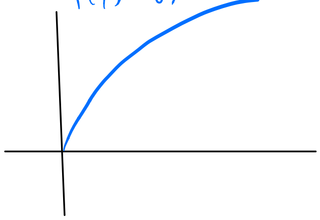


$f(x) = \cos x$ $\quad$ $f(x) = \cos 2x$ $\quad$ $f(x) = \cos \frac{x}{2}$

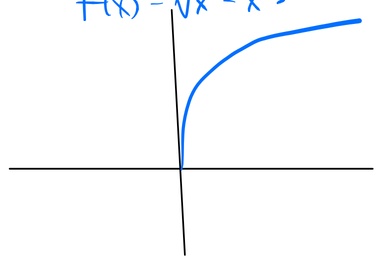


Others:

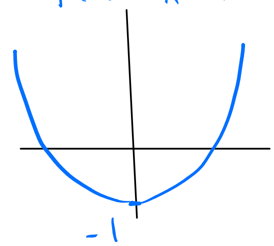
$f(x) = \sqrt{x} = x^{1/2}$



$f(x) = \sqrt[3]{x} = x^{1/3}$



$f(x) = x^2 - 1$



Def: A rational function is the quotient of two polynomials.

Limits of rational functions (what to try, for $\lim_{x\to a} f(x)$)

1. Plug in $x = a$
2. Factor top & bottom, e.g., $\lim_{x\to 1} \frac{x-1}{(x^2-6x+5)} = \lim_{x\to 1} \frac{(x-1)}{(x-1)(x-5)} = \frac{-1}{4}$
3. Graph it, inspect, left & right-hand limits.

Limits of rational functions at ∞: "big guy over big guy".

Ex:

- $\lim_{x\to\infty} \frac{3x^5 - 4x^2 + 6}{4x^5 + 10x} = \lim_{x\to\infty} \frac{3x^5}{4x^5} = \lim_{x\to\infty} \frac{3}{4}$
- $\lim_{x\to\infty} \frac{x+2}{x+4} = \lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} 1 = 1$
- $\lim_{x\to\infty} \frac{8x^3 - 2}{x^4 + 2x^2} = \lim_{x\to\infty} \frac{8x^3}{x^4} = \lim_{x\to\infty} \frac{8}{x} = 0.$
- $\lim_{x\to\infty} \frac{5x^6 + 3x - 1}{4x^4 + 2} = \lim_{x\to\infty} \frac{5x^6}{4x^4} = \lim_{x\to\infty} \frac{5}{4}x^2 = \infty$ (DNE)
- $\lim_{x\to\infty} \frac{\sqrt{9x^6 + 2x}}{2x^3 + 1} = \lim_{x\to\infty} \frac{\sqrt{9x^6}}{2x^3} = \lim_{x\to\infty} \frac{3x^3}{2x^3} = \lim_{x\to\infty} \frac{3}{2} = \frac{3}{2}$

Note: $\sqrt{9x^6} = \sqrt{9}\sqrt{x^6} = 3 \cdot (x^6)^{1/2} = 3x^3$

★ Think of $\lim_{x \to \infty} f(x)$ as what we get if we "plug in ∞"

Ex:

- $\lim_{x \to \infty} \frac{3}{e^x + 3x}$ "=" $\frac{3}{\infty} = 0$
- $\lim_{x \to \infty} \frac{\sin x}{x} = \frac{?}{\infty} = 0$
- $\lim_{x \to \infty} \frac{3x}{e^x} = ???$ [Intuition: 0 because e^x "grows faster"]

We'll learn how to do this later... [need calculus]

Fri 9/6

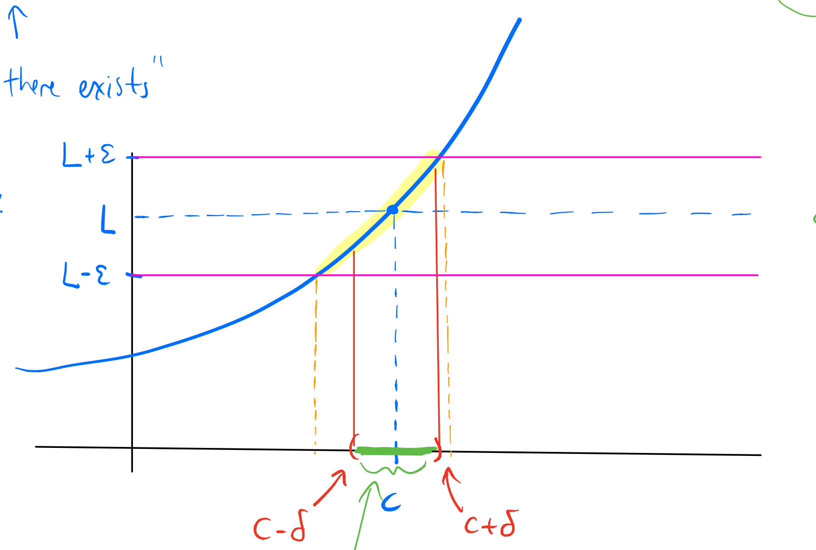
The formal definition of a limit: $\lim_{x \to c} f(x) = L$ means:

$\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

$\forall$ "for all"

$\exists$ "there exists"

Picture of this:



If the input x is within this range, then the output will be within ε of L.