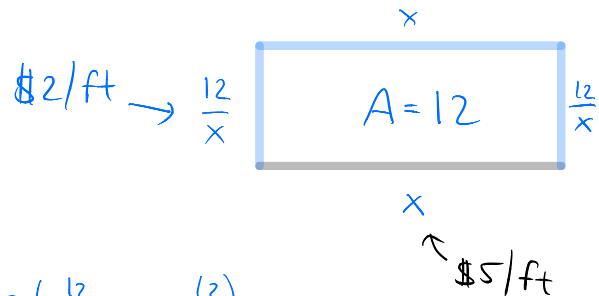
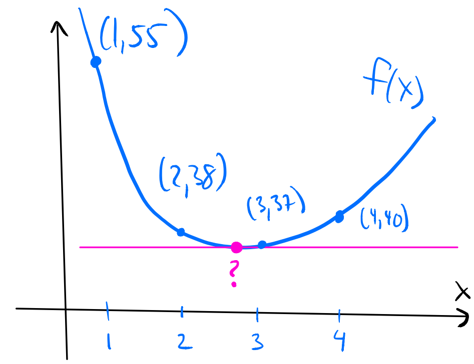
Fri. 9/6

<u>Motivating example</u> (recall):



The cost to build a fence around a 12 ft^2 region is $c(x) = \$5x + \$2\left(\frac{12}{x} + x + \frac{12}{x}\right)$

$$= 7x + \frac{24}{x}.$$

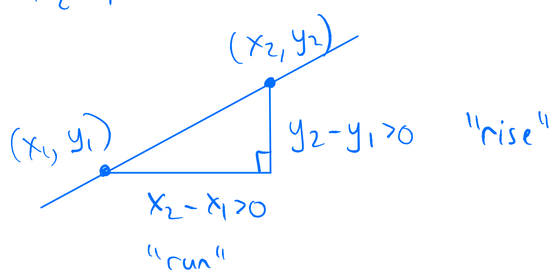


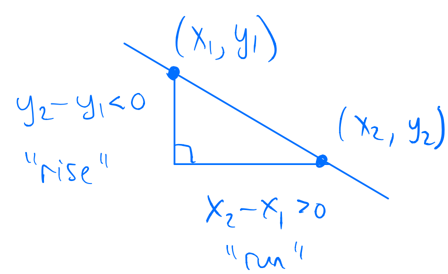
<u>Goal</u>: How to <u>minimize</u> $c(x)$.

<u>Key observation</u>: The minimum is where the "tangent line" is <u>horizontal</u>, i.e., it has slope $m = 0$

<u>Recall</u>: The <u>slope</u> of the line through (x_1, y_1) and (x_2, y_2) is

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$. "rise over run"





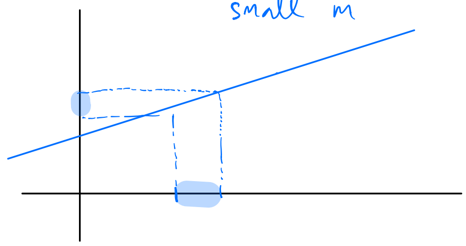
positive slope

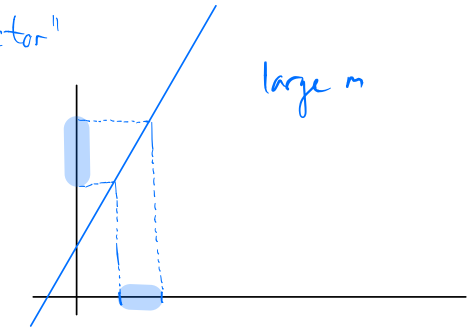
the line is <u>rising</u>

negative slope

the line is <u>falling</u>.

Think of the slope as a "stretching factor"





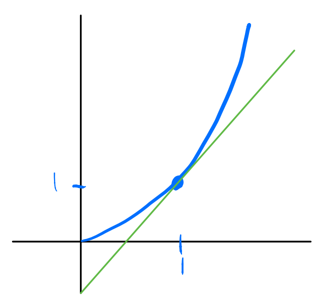
Mon 9/9

Average velocity Let $x(t)$ be the position of a car at time t. Then the average velocity (rate of change) of $x(t)$ b/w $t=a$ and $t=b$ is

$$\frac{x(b)-x(a)}{b-a}.$$

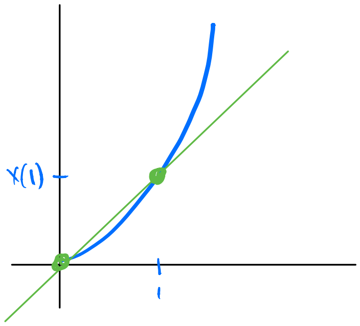
Think about how average velocity compares to instantaneous velocity.

Example: Suppose $x(t)=t^2$, for $0 \le t \le 10$.
What is the instantaneous velocity at $t=1$?



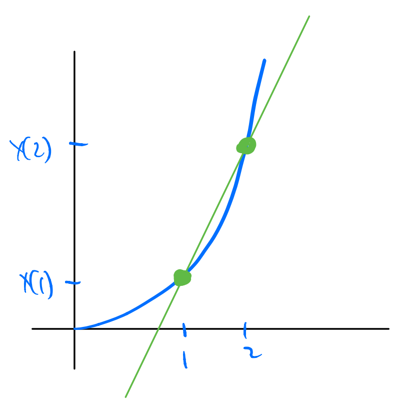
right answer
(slope of this line)

Wrong answers



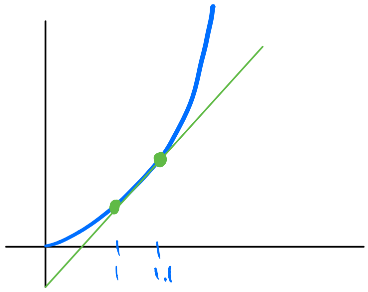
$$\frac{x(1)-x(0)}{1-0} = \frac{1^2-0^2}{1} = 1$$

ave. vel. b/w $t=0$ & $t=1$



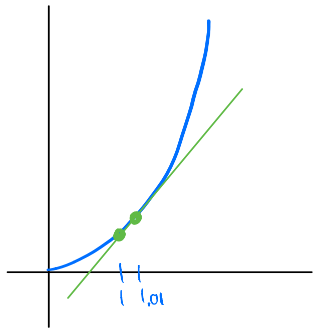
$$\frac{x(2)-x(1)}{2-1} = \frac{2^2-1^2}{1} = 3$$

ave. vel. b/w $t=1$ & $t=2$



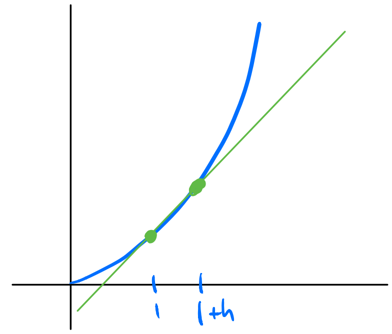
$$\frac{x(1.1)-x(1)}{1.1-1} = \frac{1.1^2-1^2}{0.1} = 2.1$$

ave. vel. b/w $t=1$ & $t=1.1$



$$\frac{x(1.01)-x(1)}{1.01-1} = \frac{1.01^2-1^2}{0.1} = 2.01$$

ave. vel. b/w $t=1$ & $t=1.01$



$$\frac{x(1+h)-x(1)}{(1+h)-1} = \frac{(1+h)^2-1^2}{h}$$

$$= \frac{1+2h+h^2-1}{2} = \boxed{2+h}$$

<u>Note</u>: The <u>instantaneous</u> rate of change at $t=1$ is $\lim_{h\to 0}(2+h) = 2$.

<u>Def</u>: The instantaneous rate of change of $f(x)$ at $x=c$ is the function $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, called the <u>derivative</u> of $f(x)$, denoted $f'(x)$.

★ $f'(x)$ = "slope of the tangent line to f at $(x, f(x))$".

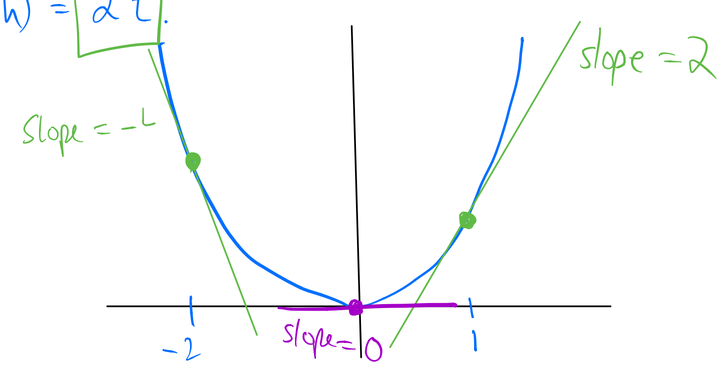
<u>Ex</u>: Consider the same function above, $f(t) = t^2$.

Compute the derivative of $f(t)$ at an arbitrary point t.

<u>Ans</u>: $$\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{(t+h)^2-t^2}{h} = \lim_{h\to 0}\frac{\cancel{t^2}+2ht+h^2-\cancel{t^2}}{h}$$

$$= \lim_{h\to 0}\left(\frac{2ht}{h}+\frac{h^2}{h}\right) = \lim_{h\to 0}(2t+h) = \boxed{2t}.$$

<u>What this means</u>:



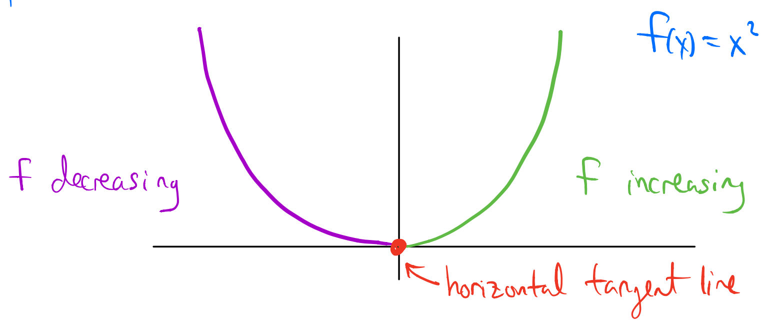
Wed 9/11

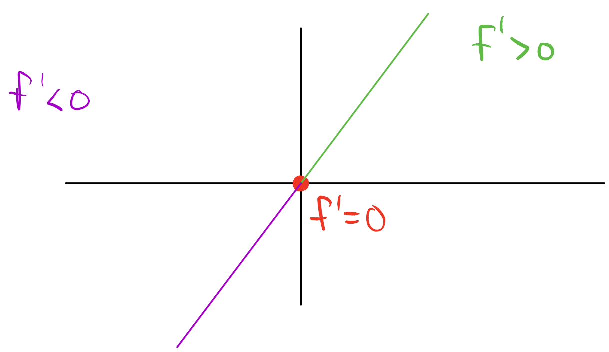
The interplay between a function and its derivatives

Note: $f'(x) > 0$ means $f(x)$ is increasing, or "rising"

$f'(x) < 0$ means $f(x)$ is decreasing, or "falling"

Ex: Compare $f(x) = x^2$ vs. $f'(x) = 2x$.



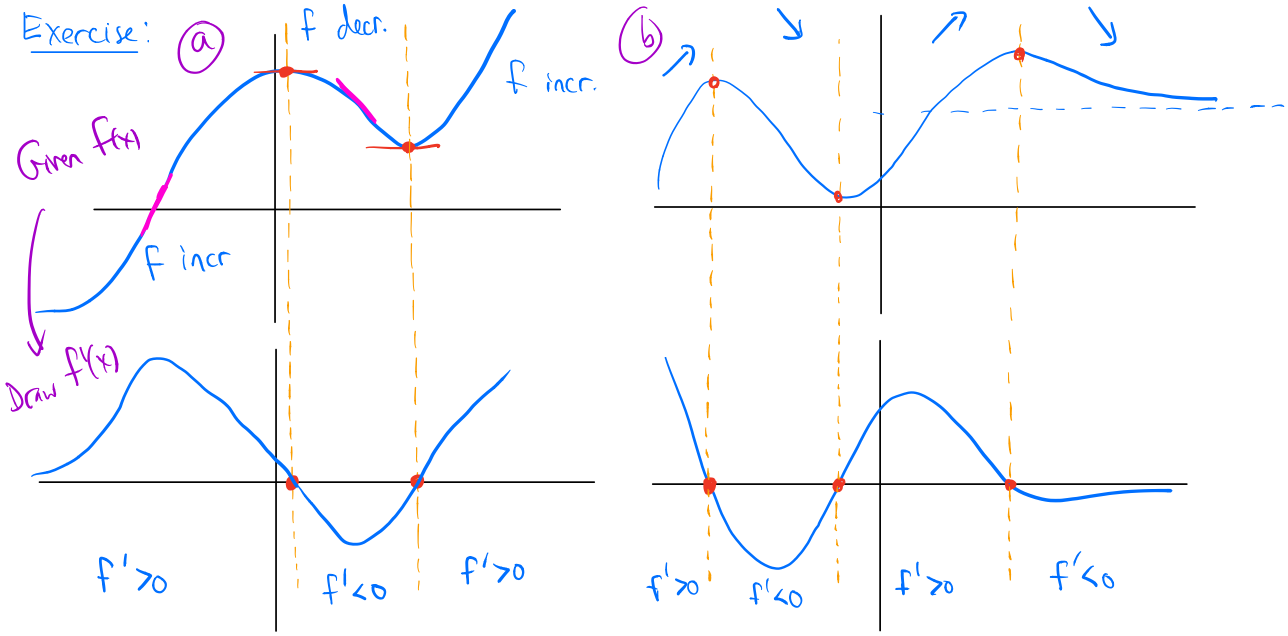


Def: If the tangent line at $(c, f(c))$ is horizontal, then c is a critical point.

Exercise: 1. Given a function $f(x)$, graph its derivative, $f'(x)$.

2. Given the derivative $f'(x)$, graph the function.

Exercise:



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The second derivative:

Motivation: Let $x(t)$ = position of a car (or ball, etc.) at time t

Then $x'(t)$ = rate of change of position

= velocity = $v(t)$

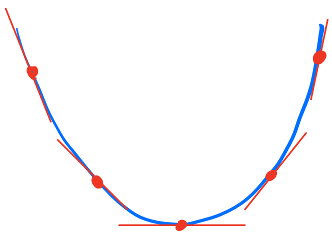
★ What is $v'(t)$? "rate of change of velocity"?

Ans: "acceleration".

We call this the second derivative of $f(x)$, the "rate of change of the rate of change," denoted $f''(x)$.

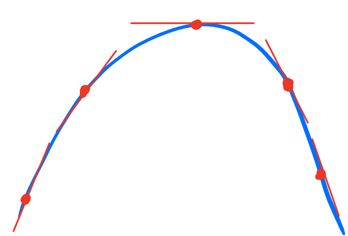
What does the second derivative measure?

$f'' > 0$ — slope of tangent line is increasing



"concave up"

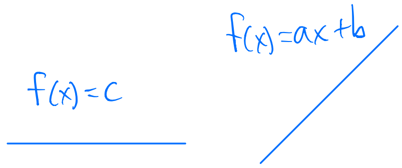
$f'' < 0$ — slope of tangent line is decreasing

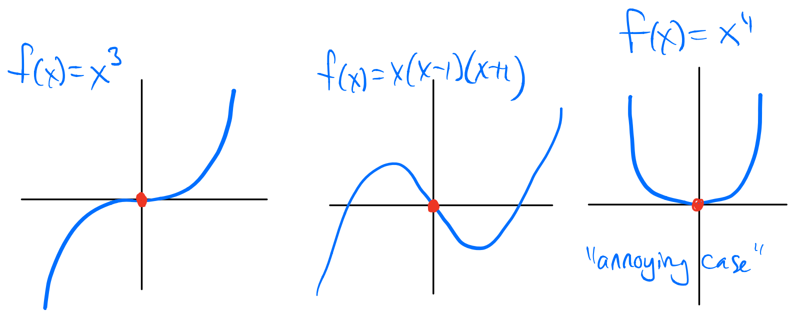


"concave down"

what about $f''=0$?

Ex:





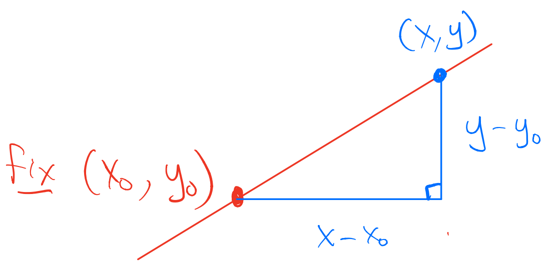
☆ Key takeaway: If $f''=0$, we don't necessarily know about the concavity.

Tangent line approximation

Goals: (1) What is the formula for the tangent line approximation to $f(x)$ at $x=a$?

(2) How can we use this to approximate $f(x)$ for $x \approx a$?

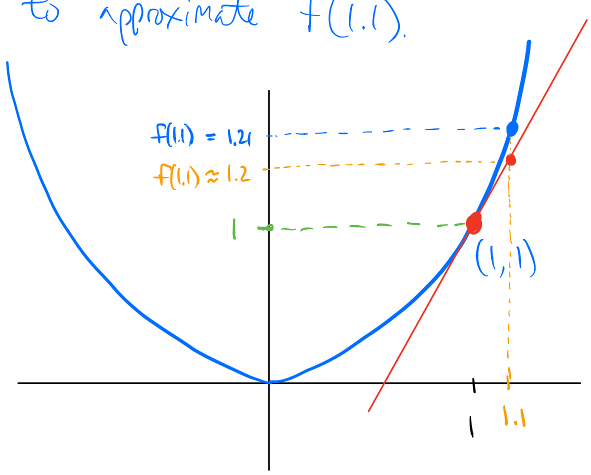
Recall: Formula for a line through (x_0, y_0) with slope m:



slope $m = \dfrac{y-y_0}{x-x_0}$

$$\Longrightarrow \quad \boxed{y-y_0 = m(x-x_0)}$$

Exercise: Find the tangent line approximation to $f(x)=x^2$ at $x=1$. Use this to approximate $f(1.1)$.



Recall: $f'(x)=2x$, so $f'(1)=2$

Here: $(x_0, y_0) = (1,1)$

$m = f'(1) =$

So $y-y_0 = m(x-x_0)$

is $y-1 = 2(x-1)$

$$\Rightarrow \boxed{y = 2x - 1}$$

Now, use this to approximate $f(1.1)$.

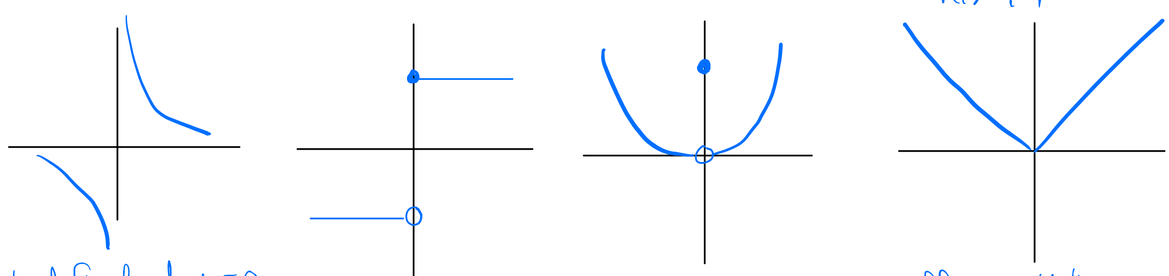
$f(1.1) \approx 2(1.1) - 1 = 1.2$. Actual value: $f(1.1) = 1.1^2 = 1.21$

This only works for $x \approx 1$.

Mon 9/16

Sometimes there is no linear approximation. This happens when you zoom in, the function doesn't look like a straight line.

<u>Ex:</u> $f(x) = |x|$



not defined at $x=0$ — not continuous at $x=0$ — not "differentiable" at $x=0$.

☆ How to characterize this?

$f(x)$ is: <u>continuous</u> at $x=a$ if: $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

<u>differentiable</u> at $x=a$ if: (1) $f(x)$ is continuous at $x=a$

and (2) $\lim_{x \to a^-} f'(x) = \lim_{x \to a^+} f'(x)$

<u>Non-example</u>; how (2) could hold but (1) could fail: