Mon 9/16

Notation: Write $(f(x))'$ for $f'(x)$.

Derivatives of polynomials

- $f(x) = x^n$, where $n = 0, 1, 2, \ldots$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\left[\cancel{x^n} + n x^{n-1} h + \_\_ x^{n-2} h^2 + \cdots + h^n\right] - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} n x^{n-1} + h[\quad \ldots \quad] = n x^{n-1}$$

- Derivative of sums:

$$(f+g)' = \lim_{h\to 0} \frac{[f(x+h) - g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \quad f'(x) \quad + \quad g'(x)$$

- Derivatives & scalar multiplication:

$$(cf)' = \lim_{h\to 0} \frac{c f(x+h) - c f(x+h)}{h}$$

$$= \lim_{h\to 0} c \left[\frac{f(x+h) - f(x)}{h}\right] = c \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = c \cdot f'(x).$$

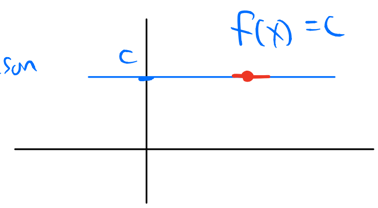
☆ Now we can differentiate polynomials.

Example: $(x^5 + 4x^3 - 6x - 2)' = 5x^4 + 12x^2 - 6$.

Remark: If $f(x) = c$, then $f'(x) = 0$.

Why: ① Algebraic reason: $c = c \cdot x^0 \Rightarrow f'(x) = c \cdot 0 x^{-1} = 0$.

② Graphical reason



Slope of the tangent line is zero everywhere.

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- Reciprocal rule:

$$\left(\frac{1}{f}\right)' = \lim_{h\to 0} \frac{\left[\frac{1}{f(x+h)} - \frac{1}{f(x)}\right]}{h} = \lim_{h\to 0}\left[\frac{f(x)}{f(x+h)\,f(x)} - \frac{f(x+h)}{f(x+h)\,f(x)}\right]\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{f(x)-f(x+h)}{f(x+h)\,f(x)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-[f(x+h)-f(x)]}{h} \cdot \frac{1}{f(x+h)\,f(x)}$$

$$= -\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} \frac{1}{f(x+h)\,f(x)}$$

$$= \quad - f'(x) \quad \cdot \quad \frac{1}{(f(x))^2} \quad = \quad \frac{-f'(x)}{(f(x))^2}$$

Example: Compute $(x^{-3})' = \left(\frac{1}{x^3}\right)'$

Let $f(x) = x^3$, so $f'(x) = 3x^2$.

$$(x^{-3})' = \left(\frac{1}{f}\right)' = \frac{-f'(x)}{(f(x))^2} = \frac{-3x^2}{(x^3)^2} = \frac{-3x^2}{x^6} = \boxed{-3x^{-4} = \frac{-3}{x^4}}.$$

This generalizes further:

Example: Compute $(x^{-n})' = \left(\frac{1}{x^n}\right)'$.

Let $f(x) = x^n$, so $f'(x) = n\,x^{n-1}$

$$(x^{-n})' = \left(\frac{1}{f}\right)' = \frac{-f'(x)}{(f(x))^2} = \frac{-n\,x^{n-1}}{(x^n)^2} = \frac{-n\,x^{n-1}}{x^{2n}} = \boxed{-n\,x^{-n-1} = \frac{-n}{x^{n+1}}}$$

<u>Remark</u>: The formula $(x^n)' = n x^{n-1}$ actually holds for <u>all</u> integers n.

- <u>Product rule</u>: Note that $(f \cdot g)' \neq f' \cdot g'$!

$$(fg)' = \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \cdot f(x)$$

$$f'(x) \cdot g(x) + g'(x)\, f(x)$$

- <u>Quotient rule</u> Note that $(f/g)' \neq f'/g'$!

Key point: $\frac{f}{g} = f \cdot \frac{1}{g}$. We'll use the <u>product</u> and <u>reciprocal</u> rules.

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)'$$

$$= \frac{f'}{g} + f \cdot \frac{-g'}{g^2} = \frac{f'g}{g^2} - \frac{fg'}{g^2} = \boxed{\frac{f'g - fg'}{g^2}}$$

<u>Example</u>: $$\left(\frac{x^3 - 2x^2 - 4}{x^{44} - 16}\right)' = \frac{(3x^2 - 4x)(x^{44} - 16) - (x^3 - 2x^2 - 4) \cdot 44x^{43}}{(x^{44} - 16)^2}$$

Fri: 9/20

Notation for derivatives:

Lagrange (Italy, 1736-1813): $f'(x)$

Euler (Switzerland, 1707-83): Df

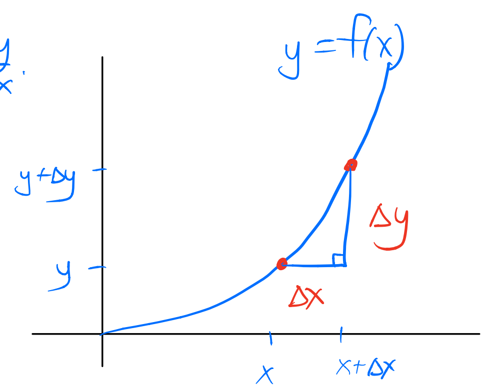
Newton (England, 1643-1727): $\dot{y}$ adopted in Britian

Leibniz (Germany, 1646-1716): $\frac{dy}{dx}$ adopted in Europe

Advantages of Leibniz's notation resulted in Britian falling behind 100-200 hundred years to mainland Europe, mathematically.

Leibniz's notation: If $y=f(x)$, then $\frac{dy}{dx}=f'(x)$

Motivating example: Slope of secant line is $\frac{\Delta y}{\Delta x}$.



$$\boxed{\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}}$$

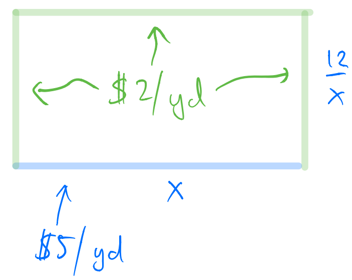
Notational reasoning: Ancient Greek symbol Δ had evolved "in the limit" to the modern "d."

Another way to interpret this: $\frac{dy}{dx}=\frac{d}{dx}(y)$.

Second derivative: $\frac{d}{dx}\left(\frac{d}{dx}(y)\right)=\frac{d^2y}{dx^2}$

Application to a problem from Day 1:

Cost of fence is $c(x) = 7x + \frac{48}{x} = 7x + 48x^{-1}$



Goal: Find min of $c(x)$.

$c'(x) = 7 - 48x^{-2} = 0$

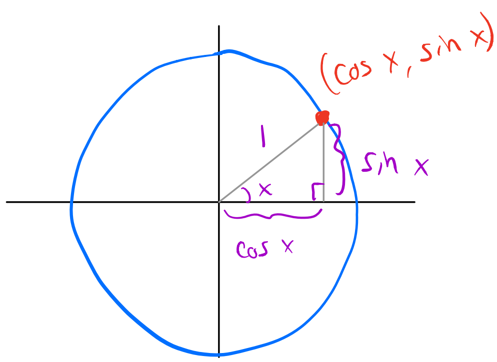
$$7 - \frac{48}{x^2} = 0 \implies x^2 = \frac{48}{7} \implies x = \pm\sqrt{\frac{48}{7}} \approx 2.619$$

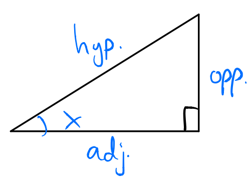
$$\implies \text{min cost is } c\left(\sqrt{48/7}\right) \approx 36.661.$$

Mon. 9/23

Next goal: Compute derivatives of trig functions.

Quick review:



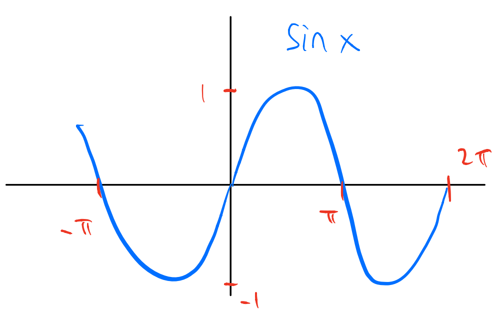


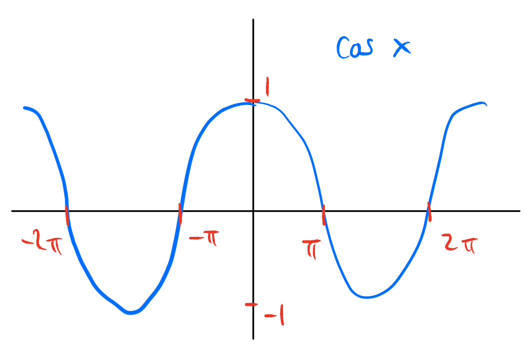
$\sin x = \frac{\text{opp}}{\text{hyp}}$

$\cos x = \frac{\text{adj.}}{\text{hyp.}}$

$\tan x = \frac{\sin x}{\cos x} = \frac{\text{opp}}{\text{adj.}}$

Also, $\frac{1}{\sin x} = \csc x$, $\frac{1}{\cos x} = \sec x$, $\frac{1}{\tan x} = \cot x$



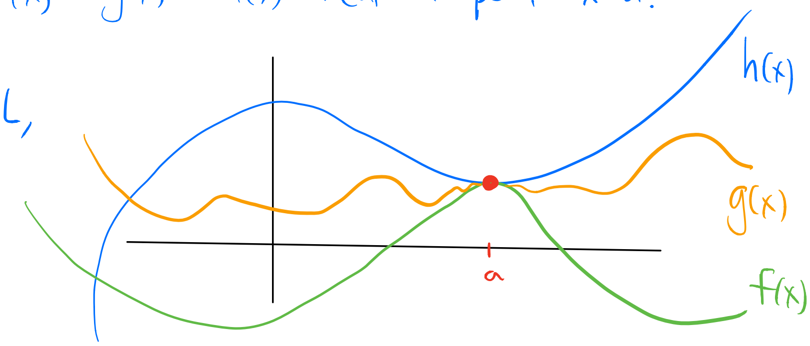


We'll need the following result to compute $\lim_{x\to 0} \frac{\sin x}{x}$ and $\lim_{x \to 0} \frac{\cos x - 1}{x}$

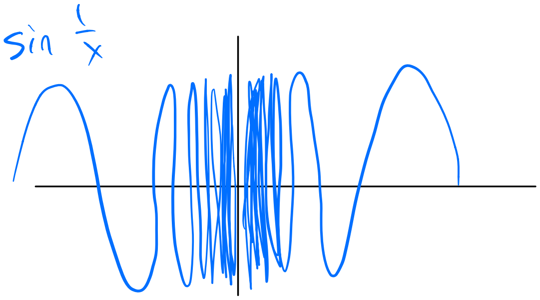
<u>Squeeze Theorem</u>: Suppose $f(x) \le g(x) \le h(x)$ near a point $x = a$.

If $\lim_{x\to a} f(x) = \lim_{x \to a} h(x) = L$,

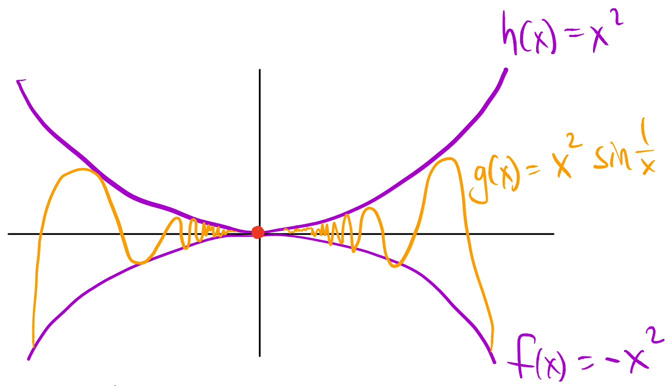
then $\lim_{x \to a} g(x) = L$



<u>Application</u>: Compute $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$



$\lim_{x\to 0} \sin \frac{1}{x}$ DNE



$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$ by Squeeze theorem

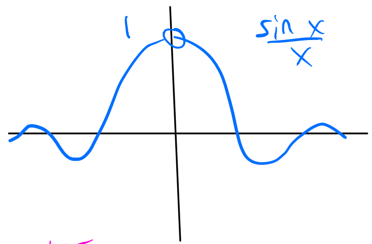
<u>Formally</u>: $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$

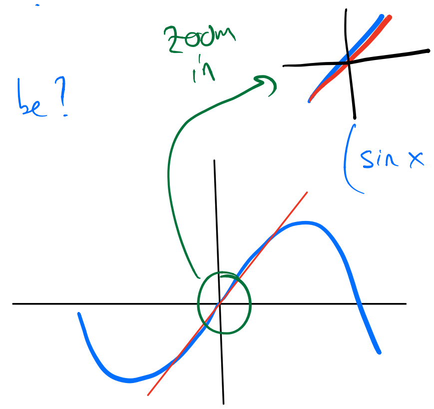
as $x \to 0$ $\downarrow$ 0 ; by Squeeze thm $\downarrow$ 0 ; as $x \to 0$ $\downarrow$ 0

<u>Tools we'll need</u>: Find $\lim_{h\to 0} \frac{\sin h}{h}$ and $\lim_{h \to 0} \frac{\cos h - 1}{h}$.

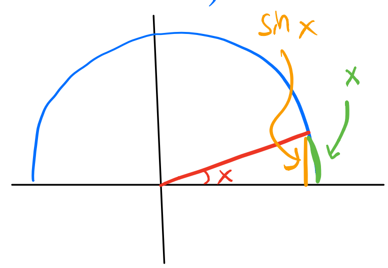
☆ How to compute $\lim\limits_{x\to 0} \frac{\sin x}{x}$?

First, what "should" it be?





($\sin x \approx x$ for $x \approx 0$)



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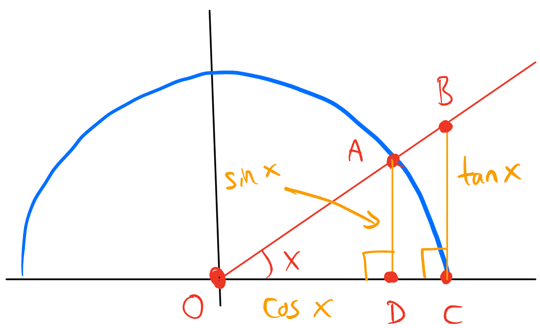
Now, let's verify this.



area ΔAOD < area sector AOC < area ΔOBC

$$\frac{1}{2}\sin x\cos x \leqslant \frac{x}{2} \leqslant \frac{1}{2}\tan x$$

(mult. by $\frac{2}{\sin x}$) $\Rightarrow \cos x \leqslant \frac{x}{\sin x} \leqslant \frac{1}{\cos x}$

(take reciprocal) $\Rightarrow \frac{1}{\cos x} \geqslant \frac{\sin x}{x} \geqslant \cos x$

$$\underbrace{\lim_{x\to 0}\frac{1}{\cos x}}_{=1} \geqslant \lim_{x\to 0}\frac{\sin x}{x} \geqslant \underbrace{\lim_{x\to 0}\cos x}_{=1}$$

Thus, $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ by the Squeeze theorem.

Exercise: Show that $\lim\limits_{x\to 0}\frac{\cos x - 1}{x} = 0$

(We'll assume this as a fact, henceforth.)

Derivative of trig functions

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{[\cos x \sin h + \sin x \cos h] - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \sin h}{h} + \lim_{h\to 0} \frac{\sin x (\cos h - 1)}{h}$$

$$= \lim_{h\to 0} \cos x \lim_{h\to 0} \frac{\sin h}{h} + \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\cos h - 1}{x}$$

$$= \cos x \cdot 1 + \sin x \cdot 0 = \boxed{\cos x}$$

$$(\cos x)' = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{[\cos x \cosh - \sin x \sin h] - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x (\cosh - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$

$$= \lim_{h\to 0} \cos x \lim_{h\to 0} \frac{\cosh - 1}{h} - \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1 = \boxed{-\sin x}$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} = \frac{1}{(\cos x)^2} = \boxed{(\sec x)^2}$$

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Derivatives of the other trig functions can be found using the quotient rule.

Summary:

$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$

$(\tan x)' = (\sec x)^2 \qquad (\cot x)' = -(\csc x)^2$

$(\sec x)' = \sec x \tan x \qquad (\csc x)' = -\csc x \cot x$

Note: The 2nd column can be gotten from the 1st column by adding or removing "co", and a negative sign.

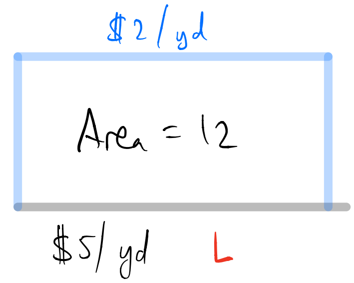
$\boxed{\text{Chain rule}}$ We'll begin by motivating the concepts, multiple ways.

Old example:

cost $C = 7L + \frac{48}{L}$ "C is a function of L"

length $L = \frac{12}{W}$ "L is a function of W"

$W = \frac{12}{L}$



$$C = 7\left(\frac{12}{W}\right) + \frac{48}{12/W}$$

$$= \frac{84}{W} + 4W \qquad \text{"C is a function of W"}$$

Question: How are the derivatives $\frac{dC}{dL}, \frac{dL}{dW}, \frac{dC}{dW}$ related?

Analogy: Suppose Clemson scores 3x as much as Tennessee

Suppose Tenn scores 4x as much as USC.

Question: How much more does Clemson score as USC?

Ans: $\dfrac{d\,\text{Clemson}}{d\,\text{USC}} = \dfrac{d\,\text{Clemson}}{d\,\text{Tenn}} \cdot \dfrac{d\,\text{Tenn}}{d\,\text{USC}}$

$= 3 \cdot 4 = 12$

The chain rule tells us how to compute the derivative of $f(g(x))$.

For example, consider $\sin 2x$.

If $f(x) = \sin x$, $g(x) = 2x$, then $f(g(x)) = \sin 2x$

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & 2x \arrow[r, "f"] & \sin 2x \end{tikzcd}$$

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Chain rule: Given $f(g(x))$,

$$[f(g(x))]' = f'(g(x)) \cdot g'(x) \qquad \frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

old notation (Lagrange) New notation (Leibnitz)

Exercises:

- $(\sin 2x)' = \dfrac{d}{dx}(\sin 2x)$

  write as $f(x) = \sin x$ $\quad f'(x) = \cos x$

  $g(x) = 2x$ $\quad g'(x) = 2$

  $f(g(x))' = f'(g(x)) \cdot g'(x) = \cos 2x \cdot 2 = 2 \cos 2x$

- $\left[(\cos x)^3\right]' = \frac{d}{dx}\left[(\cos x)^3\right]$

  Write as $f(x) = x^3$ $\quad f'(x) = 3x^2$

  $g(x) = \cos x$ $\quad g'(x) = -\sin x$

  $$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x) = 3(\cos x)^2 \cdot (-\sin x) = -3(\cos x)^2 \sin x.$$

- $\left[(6x^3+7x)^8\right]' = \frac{d}{dx}(6x^3+7x)^8$

  Let's use both notations and compare/contrast.

<u>Lagrange:</u>

$f(x) = x^8$ $\quad f'(x) = 8x^7$

$g(x) = 6x^3+7x$ $\quad g'(x) = 18x^2+7$

$$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$$
$$= 8(6x^3+7x)^7 \cdot (18x^2+7)$$

<u>Leibnitz:</u>

Let $y = u^8$, where $u = 6x^3+7x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= 8u^7 \cdot (18x^2+7)$$
$$= 8(6x^3+7x)^7 \cdot (18x^2+7)$$

<u>Useful identities</u>

$(\sin kx)' = k \cos kx$

$(\cos kx)' = -k \sin kx$

Wed 10/2

Implicit differentiation

What's the difference between defining a function "explicitly" vs. "implicitly"?

- **Functions defined explicitly**

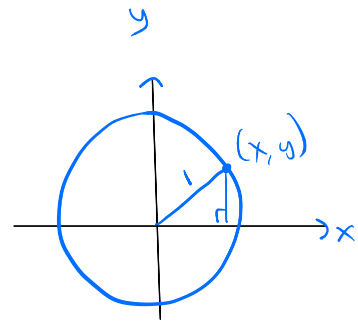
Examples: $f(x) = x^2$

$y = \sin(2x) + 3x$

$$y(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- **"Functions" defined implicitly**

Examples: $x^2 + y^2 = 1$. This is a circle:



In this case, we can actually solve for y:

$y = \pm\sqrt{1-x^2}$, so it's actually 2 functions.

But frequently, we can't solve for y.

For example, consider the curve defined by $xy + x\sin y = 3x$

☆ Key point: Even for implicitly defined functions, we can still find the derivative, $\frac{dy}{dx}$!

Method: Differentiate both sides & solve for $\frac{dy}{dx}$

$$xy + x\sin y = 3x$$

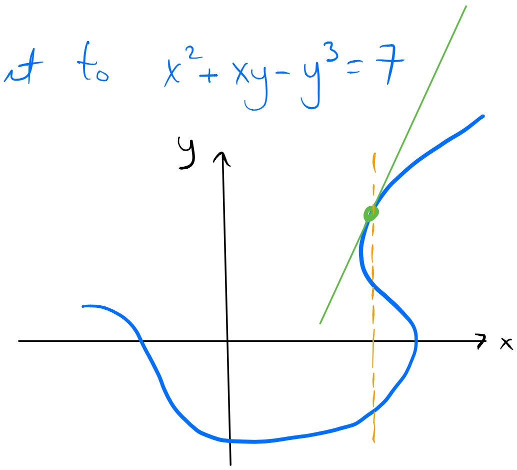
$$(xy)' + (x\sin y)' = (3x)'$$

$$\left(1\cdot y + x\cdot\frac{dy}{dx}\right) + \left(1\cdot\sin y + x\cdot\cos y\cdot\frac{dy}{dx}\right) = 3 \leftarrow \begin{cases} x \text{ is a variable, but} \\ y \text{ is a function!} \end{cases}$$

$$(x + x\cos y)\frac{dy}{dx} + y + \sin y = 3$$

$$\boxed{\frac{dy}{dx} = \frac{3-y-\sin y}{x+x\cos y}}$$

Example: Find the equation of the line tangent to $x^2+xy-y^3=7$ at the point $(x_0, y_0) = (3,2)$



Note: This is not a function (fails vertical line test), so we prefer to write $\frac{dy}{dx}$ to $y'(x)$.

$$\frac{d}{dx}\left(x^2+xy-y^3\right) = \frac{d}{dx}(7)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) - \frac{d}{dx}(y^3) = 0$$

$$2x + y + x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$2x+y = (3y^2-x)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{2x+y}{3y^2-x}} \implies \left.\frac{dy}{dx}\right|_{(x,y)=(3,2)} = \frac{2\cdot 3+2}{3\cdot 2^2-3} = \frac{8}{9}$$

Tangent line: $y - y_0 = m\,(x - x_0) \implies \boxed{y-2 = \frac{8}{9}(x-3)}$

$$\underset{2}{\underset{\uparrow}{y_0}} \qquad \underset{8/9}{\underset{\uparrow}{m}} \qquad \underset{3}{\underset{\uparrow}{x_0}}$$