Fri 10/4

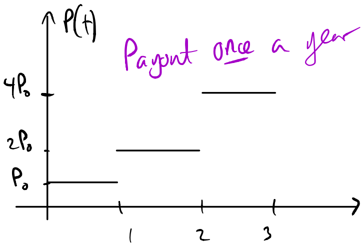
$\boxed{\text{Exponential functions}}$

We hear a lot that $e \approx 2.71828 1828\ldots$ is a number that "comes up a lot in nature."

But what does that mean?

<u>Motivating example:</u>

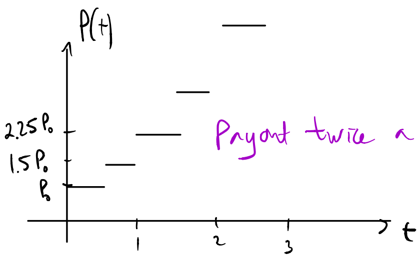
Consider an investment that grows at a 100% rate



Payout <u>once</u> a year (100%)

After 1 year: $P(1) = P_0(1+1) = 2P_0$

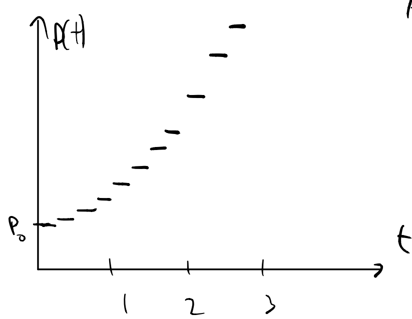
$\underset{\text{original}}{\uparrow}\ \underset{\text{gained}}{\uparrow}$



After 1 year: $P(1) = P_0\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}\right)$

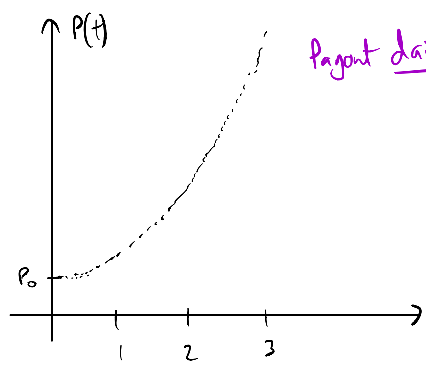
$= P_0\left(1+\frac{1}{2}\right)^2 = 2.25P_0$

Payout twice a year (50%)



After 1 year: $P(1) = P_0\left(1+\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{4}\right)$

$= P_0\left(1+\frac{1}{4}\right)^4 \approx 2.441\, P_0$

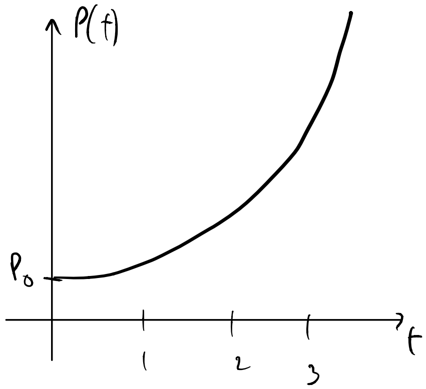


Payout <u>daily</u> $\left(\frac{100}{365}\%\right)$

After 1 year: $P(1) = P_0\left(1+\frac{1}{365}\right)^{365}$

$\approx 2.7146\, P_0$

$\star$ In the <u>limit</u>, we say interest is compounded <u>continuously</u>.



After 1 year:

$$P(1) = P_0 \cdot \underbrace{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}_{\text{call this "e", } \approx 2.718281828\ldots}$$

e was first discovered by John Napier in the early 1600's.

It arose several other times in the 1600's in different contexts

In 1683, Jacob Bernoulli showed that $e < 3$.

[Note that by our above argument, $e > 2$, $e > 2.25$, $e > 2.441$, $e > 2.7146, \ldots$]

Bernoulli: $\boxed{\text{Define } e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)}$

Note that $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + nx^{n-1} + x^n$

Plugging in $x = \frac{1}{n}$: $\left(1+\frac{1}{n}\right)^n \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 3$$

Similarly, we can define $\boxed{e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

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Derivative of e^x:

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x \cdot \boxed{\lim_{h\to 0}\frac{e^h-1}{h}}$$

Sneaky little trick to evaluate this...

Substitute: let $n = e^h - 1 \iff n+1 = e^h \iff \ln(n+1) = h$

Now, we have $\quad e^x \lim_{n\to 0} \frac{n}{\ln(n+1)} \cdot \frac{1/n}{1/n} = e^x \lim_{n\to 0} \frac{1}{\frac{1}{n}\ln(1+n)}$

$$= e^x \lim_{n\to 0} \frac{1}{\ln(1+n)^{1/n}} \qquad (\text{log laws})$$

$$= e^x \lim_{n\to\infty} \frac{1}{\ln\left(1+\frac{1}{n}\right)^n} \qquad \frac{1}{n}\to 0 \iff n\to\infty$$

$$= e^x \frac{1}{\ln\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]} \qquad \text{Swap } \lim_{n\to\infty} \text{ with } \ln$$

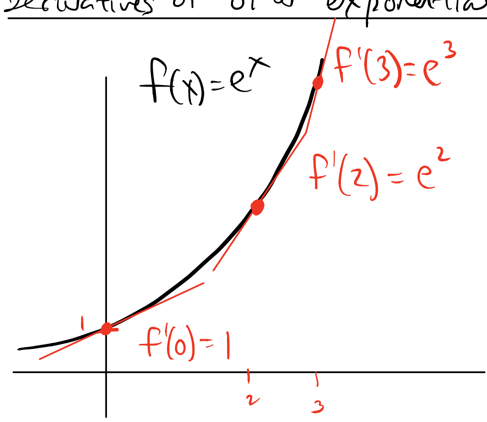
$$= e^x \frac{1}{\ln[e^x]} = e^x \cdot 1 = e^x$$

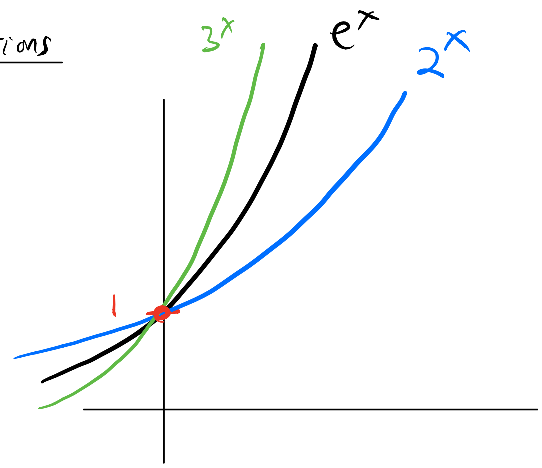
☆ Thus, $\boxed{\frac{d}{dx} e^x = e^x}$

By the chain rule, $\frac{d}{dx}(e^{kx}) = k e^{kx}$

$$\frac{d}{dx} e^{x^2} = e^{x^2} \cdot \frac{d}{dx}(x^2) = 2x e^{x^2}.$$

Derivatives of other exponential functions





Trick: $\frac{d}{dx}(2^x) = \frac{d}{dx}\left[\left(e^{\ln 2}\right)^x\right] = \frac{d}{dx}\left[e^{(\ln 2)x}\right]$ think of $\ln 2 = k \ldots$

$$= (\ln 2)\, e^{(\ln 2)x} = (\ln 2)\, 2^x$$

Similarly, $\boxed{\frac{d}{dx} b^x = (\ln b)\, b^x}$

Derivative of natural log: Use implicit differentiation

Suppose $y = \ln x$. Then $e^y = x$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$e^y \cdot \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Thus, $\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$

Fri 10/11

Related rates

We've seen some optimization (min/max) problems, e.g.,

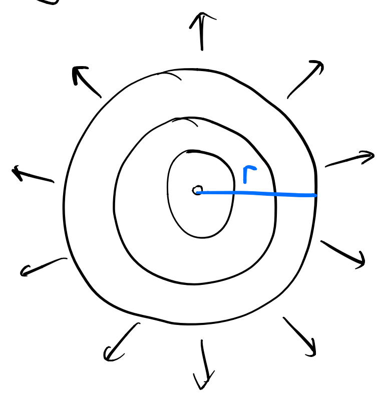
- minimize cost, subject to ...
- maximize area, subject to

In these, the process is the same: Compute $f'(x)$, set $=0$, & solve.

Now, we'll study another type of word problem: "related rates"

The process involves using the chain rule: $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Example 1: A rock is dropped in a pond, and the ripple expands at a rate of 3 in/sec. How fast is the area increasing when the radius is 7 in?



Given info: $\frac{dr}{dt} = 3 \ \text{in/sec}$

Want: $\left.\frac{dA}{dt}\right|_{r=7}$

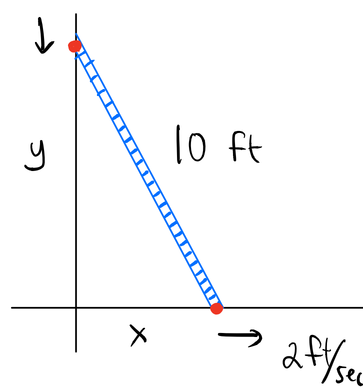
Also, $A = \pi r^2$

Chain rule: $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ $\qquad \frac{dA}{dt} = 2\pi r$

$= 2\pi r \cdot 3 = 6\pi r$

So $\left.\frac{dA}{dt}\right|_{r=7} = 6\pi(7) = \boxed{42\pi \ \frac{\text{in}^2}{\text{sec}}}$

Example 2: A 10-ft ladder rests against a wall. If the base is pulled away at a rate of 2 ft/sec, how fast is the top of the ladder falling when the ladder is 6 ft from the wall?

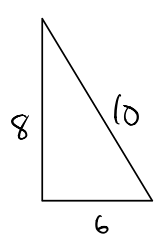


Given info: $\frac{dx}{dt} = 2 \ \text{ft/sec}$

Want $\left.\frac{dy}{dt}\right|_{x=6}$

Also, $x^2 + y^2 = 100$

Note: When $x = 6$: $6^2 + y^2 = 10^2$



$\Rightarrow y = \sqrt{100-36} = 8.$

There are often multiple ways to proceed on these types of problems.

<u>One option</u>: $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx} 100$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Now plug into: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$$= \frac{-x}{y} \cdot 2 = \frac{-6}{8} \cdot 2 = \boxed{-1.5 \frac{ft}{sec}}$$

<u>Another option</u>: $\frac{d}{dt}(x^2+y^2) = \frac{d}{dt} 100$

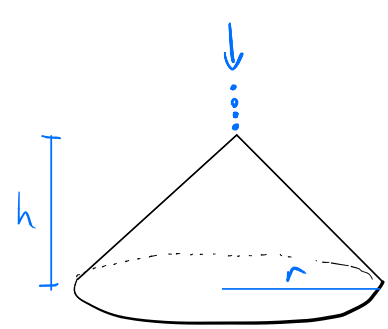
$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$2 \cdot 6 \cdot 2 + 2 \cdot 8 \cdot \frac{dy}{dt} = 0 \quad \Rightarrow \quad 16 \frac{dy}{dt} = -24$$

$$\Rightarrow \quad \frac{dy}{dt} = \boxed{-1.5 \frac{ft}{sec}}$$

<u>Moral</u>: There is often more than one way to get the right answer.

<u>Example 3</u>: Sand falls from an overhead bin. It forms a sandpile with radius that is 3 times its height.
If it falls at a rate of $120\ ft^3/min$, how fast is the height changing when the pile is 10 ft high?



$$\boxed{\begin{aligned}&\underline{\text{Given info:}}\quad r = 3h\\&\qquad V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi (3h)^2 \cdot h = 3\pi h^3.\\&\qquad \frac{dV}{dt} = 120\ ft^3/min\\&\qquad \underline{\text{Want:}}\quad \left.\frac{dh}{dt}\right|_{h=10}\end{aligned}}$$

We can also see that $V = 3\pi h^3 \Rightarrow \frac{dV}{dh} = 9\pi h^2$

Apply the chain rule to $\frac{dV}{dt}$, $\frac{dV}{dh}$, $\frac{dh}{dt}$:

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$120 = 9\pi h^2 \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{120}{9\pi h^2}$$

$$\Rightarrow \left.\frac{dh}{dt}\right|_{h=10} = \frac{120}{9\pi \cdot 100} = \boxed{\frac{12}{90\pi}\ ft/min}$$