<u>Notation</u>: If $F'(x) = f(x)$, i.e., if $F(x)$ is an antiderivative of $f(x)$, then we write $\int f(x)\,dx = F(x) + C$. We call this an indefinite integral.

In contrast, $\int_a^b f(x)\,dx$ is a definite integral

<u>Practice</u> with antiderivatives (indefinite integrals)

- $\int 1\,dx = x + C$
- $\int x\,dx = \frac{1}{2}x^2 + C$
- $\int x^2\,dx = \frac{1}{3}x^2 + C$

<u>General formula</u>: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $\quad n \neq -1$

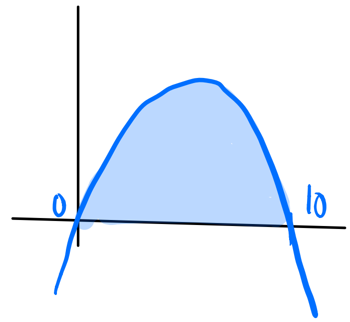
- $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C = \frac{2}{3}\sqrt{x^3} + C$
- $\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln x + C$ [<u>Note</u>: $\frac{x^0}{0}$ is undefined]

- $\int e^x\,dx = e^x + C$ $\qquad$ • $\int e^{kx} = \frac{1}{k}e^{kx} + C$
- $\int \sin x\,dx = -\cos x + C$ $\qquad$ • $\int \sin kx\,dx = -\frac{1}{k}\cos kx + C$
- $\int \cos x\,dx = \sin x + C$ $\qquad$ • $\int \cos kx\,dx = \frac{1}{k}\sin kx + C$

2

Practice with definite integrals

- $\displaystyle\int_0^{10} 60x - 6x^2 = \left(\frac{60x^2}{2} - \frac{6x^3}{3}\right)\Bigg|_0^{10}$

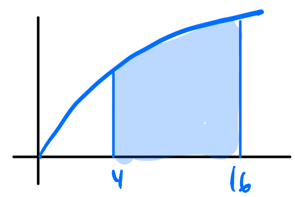


$$= (30x^2 - 2x^3)\Big|_0^{10}$$

$$= (30\cdot 10^2 - 2\cdot 10^3) - (30\cdot 0^2 - 2\cdot 0^3)$$

$$= 3000 - 2000 = \boxed{1000}$$

- $\displaystyle\int_4^{16} 3\sqrt{x}\,dx = \int_4^{16} 3x^{1/2}\,dx = \frac{3x^{3/2}}{3/2}\Bigg|_4^{16}$

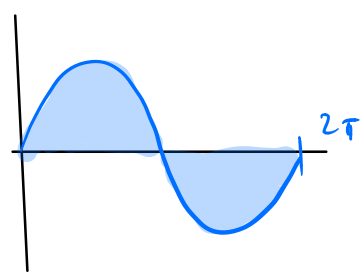


$$= 2x^{3/2}\Big|_4^{16} = 2\sqrt{x^3}\Big|_4^{16} = 2\sqrt{16^3} - 2\sqrt{4^3} = 2\cdot 64 - 2\cdot 8 = \boxed{112}$$

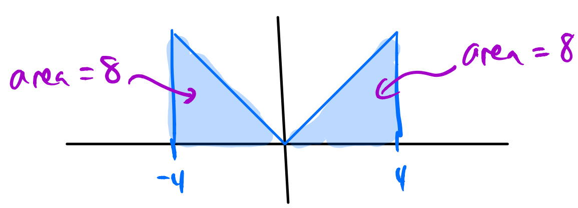
- $\displaystyle\int_0^{2\pi} 3\sin x\,dx = -3\cos x\Big|_0^{2\pi}$

$$= (-3\cos 2\pi) - (-3\cos 0)$$

$$= -3\cdot 1 + 3\cdot 1 = \boxed{0}$$



- $\displaystyle\int_{-4}^{4} |x|\,dx = 16$ (see graph)



<u>OR</u> $\displaystyle = \int_{-4}^{0} -x\,dx + \int_0^4 x\,dx$

$$= -\frac{x^2}{2}\Big|_{-4}^{0} + \frac{x^2}{2}\Big|_0^4$$

$$= -(0-8) + (8-0) = \boxed{16}$$

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Application: Average value of a function.

Motivating example: through 8 games,

Clemson	8 wins
Oklahoma	7 wins
USC	3 wins

The average # wins $= \dfrac{8+7+3}{3} = \dfrac{18}{3} = 6$ wins.

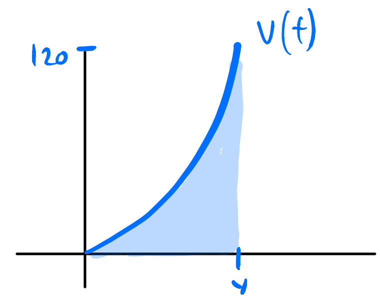
Now, consider a continuous quantity, $f(x)$.

The average value of $f(x)$ on $a \le x \le b$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

Ex 1: Suppose the velocity of a car is given by $v(t) = 30t^2$ for $0 \le t \le 2$. Find its average velocity.

Ans: $\dfrac{1}{2-0}\displaystyle\int_0^2 30t^2\,dt = \dfrac{1}{2}\,\dfrac{30t^3}{3}\Big|_0^2$

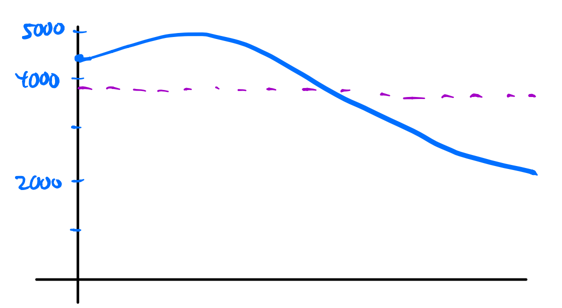
$= 5t^3\Big|_0^2 = 5\cdot 2^3 - 5\cdot 0 = 40$



Ex 2: Suppose the elevation on a 5-mile hike is

$f(x) = 60x^3 - 650x^2 + 1200x + 4500$ ft.

Find the average elevation.



Ans:

$\dfrac{1}{5-0}\displaystyle\int_0^5 60x^3 - 650x^2 + 1200x + 4500\,dx$

$= \dfrac{1}{5}\left(15x^4 - \dfrac{650x^3}{3} + 600x^2 + 4500x\right)\Big|_0^5$

$$= \frac{1}{5}\left(15 \cdot 5^4 - \frac{650}{3} \cdot 5^3 + 600 \cdot 5^2 + 4500 \cdot 5\right) \approx \boxed{3958.3 \text{ ft}}$$

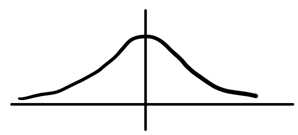
Derivatives vs. integrals

Derivatives have formulas: product, quotient, chain rules.

So we can always take the derivative of, e.g., $e^{\left[\sin\left(\frac{x^2}{x+1}\right) \cdot \cos x\right]}$.

Integrals have no such formula.

For example, $\int e^{-x^2}\,dx$ has no "closed form."



Computing integrals is more of an "art."

But there are some rules & "tricks." We'll learn one now.

u-substitution "like the chain rule in reverse"

Recall the chain rule: $\frac{d}{dx}\Big(F(g(x))\Big) = f(g(x)) \cdot g'(x)$ [where $F' = f$]

Now integrate both sides: $F(g(x)) + C = \int \underbrace{f(g(x))}_{du} \cdot \underbrace{g'(x)\,dx}_{du}$

Now let $u = g(x)$

$\Rightarrow \frac{du}{dx} = g'(x)$

$\Rightarrow du = g'(x)\,dx$

$$\boxed{F(u) + C = \int f(u)\,du}$$

Examples

① $\int 2(2x+1)^{30}\,dx$ Let $u = 2x+1 \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2\,dx$

$= \int u^{30}\,du = \frac{1}{31}u^{31} + C = \frac{1}{3}(2x+1)^{31} + C$

② $\int 3e^{10x}\,dx$ Let $u=10x \Rightarrow \frac{du}{dx}=10 \Rightarrow du=10\,dx$

$=\int e^u \cdot \frac{3}{10}\,du$ $\Rightarrow \frac{3}{10}\,du = 3\,dx$

$= \frac{3}{10}e^u + C = \frac{3}{10}e^{10x}+C.$

③ $\int x^4(x^5+6)^9\,dx$ Let $u=x^5+6$, $\frac{du}{dx}=5x^4 \Rightarrow du = 5x^4\,dx$

$=\int u^9 \cdot \frac{1}{5}\,du$ $\Rightarrow \frac{1}{5}du = x^4\,dx$

$= \frac{1}{5}\cdot\frac{u^{10}}{10}+C = \frac{1}{50}u^{10}+C = \frac{1}{50}(x^5+6)^{10}+C$

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④ $\int (\cos x)^3 \sin dx.$ Let $u=\cos x \Rightarrow du = -\sin x\,dx$

$=\int u^3(-du) = -\frac{u^4}{4}+C = \frac{(\cos x)^4}{4}+C$

⑤ $\int \frac{x\,dx}{\sqrt{1+x^2}}.$ Let $u=1+x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$=\int \frac{1}{2}u^{-1/2}\,du = \frac{1}{2}\frac{u^{1/2}}{1/2}+C = u^{1/2}+C = \sqrt{1+x^2}+C$

⑥ $\int 3x^2\sqrt{x^3+1}\,dx$ Let $u=x^3+1 \Rightarrow du = 3x^2\,dx$

$=\int u^{1/2}\,du = \frac{u^{3/2}}{3/2}+C = \frac{2}{3}\sqrt{u^3}+C = \frac{2}{3}\sqrt{(x^3+1)^3}+C$

⑦ $\int \frac{\sin t}{2-\cos t}\,dt.$ Let $u=2-\cos t \Rightarrow du = -\sin t\,dt$

$=\int -\frac{1}{u}\,du = -\ln|u|+C = -\ln(2-\cos t)+C$