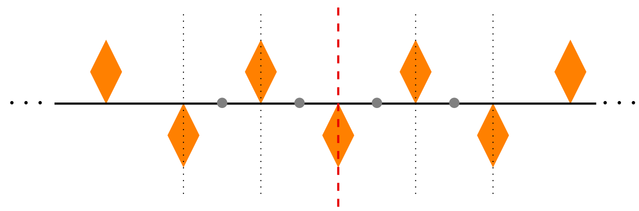
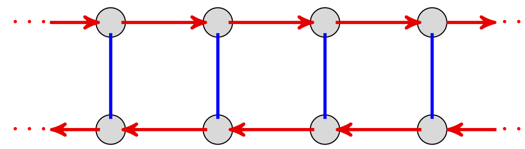
Read Chapters 3–4 of *Visual Group Theory*, or Chapters 5.1–5.3 of *IBL Abstract Algebra*. Then write up solutions to the following exercises.

1. Consider the frieze pattern shown below at left.





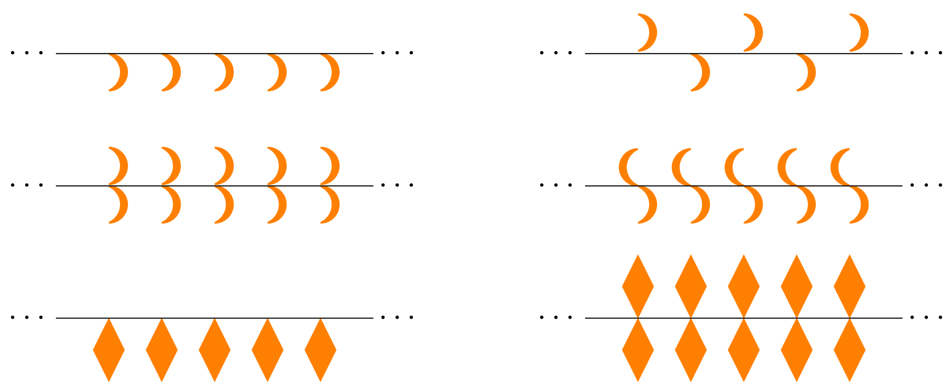
Let g be a glide-reflection to the right, and h a horizontal flip about the dashed red line. These actions (symmetries) generate a *frieze group*, and the Cayley diagram is shown on the right. Express each the following symmetries in terms of g and h:

(a) The *reflection* about each dotted line.

(b) The 180° *rotation* around each gray dot.

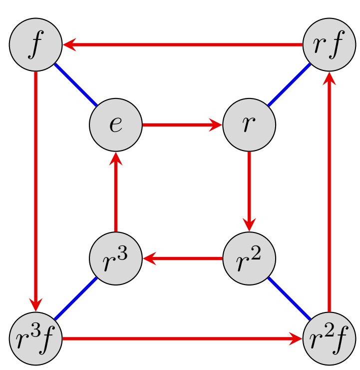
If possible (it may not be), write your answer as $g^k h g^{-k}$ for some $k \in \mathbb{Z}$.

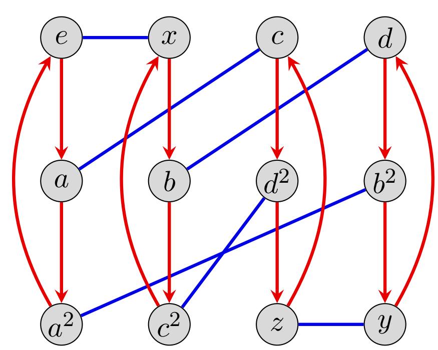
2. There are 7 freize groups. In addition to the one in the previous problem, the other 6 are symmetry groups of the friezes shown below.



For each of these friezes, draw a Cayley diagram for its frieze group using a minimal generating set. Make it clear what the generators are and write out a group presentation.

3. Shown below are the Cayley graphs of two groups: D_4 is on the left, and the right is called an "alternating group," denoted A_4.





(a) Create a multiplication table for each group. For consistency, order the elements in D_4 by $(e, r, r^2, r^3, f, rf, r^2f, r^3f)$ and by $(e, x, y, z, a, a^2, b, b^2, c, c^2, d, d^2)$ in A_4.

(b) Find the inverse of each element of each group.

(c) Write out a group presentation for each group using the generators shown in the Cayley graph.

4. In each of the following multiplication tables, let e denote the identity element. Complete each table so its depicts a group. There may be more than one way to complete a table, in which case you need to give all possibilities. Draw a Cayley diagram for each.

	e	a
e		
a		

	e	a	b
e			
a			
b			

	e	a	b	c
e				
a		e		
b				
c				

5. Show that an element cannot appear twice in the same column of a multiplication table.

6. Using our "unofficial" definition of a group, show that every group has a unique identity action e, satisfying $ge = g = eg$ for every action g in G. [*Hint*: You need to show both existence and uniqueness. For the latter, assume that e and f are both identity actions. Can you show that $e = f$?]

7. Let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the set of integers, rational numbers, and real numbers, respectively. Let $\mathbb{Z}^+$, $\mathbb{Q}^+$, and $\mathbb{R}^+$ denote the positive integers, rationals, and reals. Let $\mathbb{Z}^*$, $\mathbb{Q}^*$, and $\mathbb{R}^*$ denote the nonzero integers, rationals, and reals.

(a) Which the above sets are groups under addition? For each one that is a group, give a minimal generating set if there is one. For each one that is not, give an explicit reason for why it fails.

(b) Which of the above sets are groups under multiplication? For each one that is a group, give a minimal generating set if there is one. For each one that is not, give an explicit reason for why it fails.

(c) Let $n\mathbb{Z}$ denote the set of all integers that are multiples of n. For what $n \in \mathbb{N}$ is the set $n\mathbb{Z}$ a group under addition? Give a minimal generating set for each one that is a group.