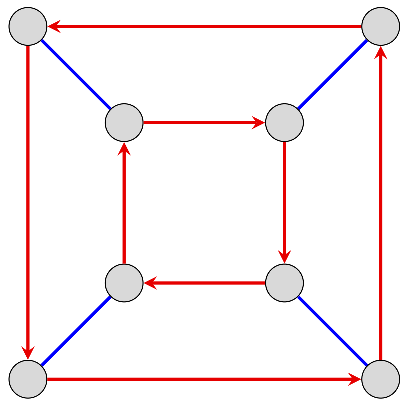
Read Chapter 6 of *Visual Group Theory*, or Chapters 4.1, 5.4, 5.5, 7 of *IBL Abstract Algebra*, and then write up solutions to the following exercises.

1. A Cayley diagram and multiplication table for the dihedral group D_4 are shown below.



	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

Section 2 of the class lecture notes describes two algorithms for expressing a group G of order n as a set of permutations in S_n. One algorithm uses the Cayley diagram and the other uses the multiplication table. In this problem, you will explore this a bit further.

(a) Label the vertices of the Cayley diagram from the set $\{1, \ldots, 8\}$ and use this to construct a permutation group isomorphic to D_4, and sitting inside S_8.

(b) Label the entries of the multiplication table from the set $\{1, \ldots, 8\}$ and use this to construct a permutation group isomorphic to D_4, and sitting inside S_8.

(c) Are the two groups you got in Parts (a) and (b) the same? (The answer will depend on your choice of labeling.) If "yes", then repeat Part (a) with a different labeling to yield a different group. If "no", then repeat Part (a) with a different labeling to yield the group you got in Part (b).

2. Find all subgroups of the following groups, and arrange them in a Hasse diagram, or subgroup lattice. Moreover, label each edge between $K \leq H$ with the index, $[H : K]$.

(a) $C_{23} = \langle r \mid r^{23} = 1 \rangle$;

(b) $C_{24} = \langle r \mid r^{24} = 1 \rangle$;

(c) $\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(a, b) \mid a, b \in \{0, 1, 2\}\}$;

(d) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(a, b, c) \mid a, b, c \in \{0, 1\}\}$; (*Tip*: it's notationally easier to write elements as binary strings, e.g., abc instead of (a, b, c));

(e) $S_3 = \{e, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$;

(f) $A_4 = \{e, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$;

(g) $Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$.

3. For each subgroup H of S_4 described below, write out all of its elements and determine what well-known group it is isomorphic to.

   (a) $H = \langle (1\,2)\,,\,(3\,4) \rangle$;

   (b) $H = \langle (1\,2)\,(3\,4)\,,\,(1\,3)\,(2\,4) \rangle$;

   (c) $H = \langle (1\,2)\,,\,(2\,3) \rangle$;

   (d) $H = \langle (1\,2)\,,\,(1\,3\,2\,4) \rangle$;

   (e) $H = \langle (1\,2\,3)\,,\,(2\,3\,4) \rangle$.

4. Prove the following, algebraically (that is, do not refer to Cayley diagrams):

   (a) If $\mathcal{H}$ is a collection of subgroups of G, then $\displaystyle\bigcap_{H_\alpha \in \mathcal{H}} H_\alpha$ is a subgroup of G.

   (b) For any (possibly infinite) subset $S \subseteq G$, the subgroup generated by S is defined as

   $$\langle S \rangle := \left\{ s_1^{e_1} s_2^{e_2} \cdots s_k^{e_k} \mid s_i \in S,\ e_i \in \{-1, 1\} \right\}.$$

   That is, $\langle S \rangle$ consists of all finite "words" that can be written using the elements in S and their inverses. Note that the s_i's need not be distinct. Prove that

   $$\langle S \rangle = \bigcap_{S \subseteq H_\alpha \leq G} H_\alpha\,,$$

   where the intersection is taken over all subgroups of G that contain S. [*Hint*: To prove that $A = B$, you need to show that that $A \subseteq B$ and $B \subseteq A$.]

5. For a subgroup $H \leq G$ and element $x \in G$, the set $xH := \{xh \mid h \in H\}$ is a *left coset* of H.

   (a) Prove that if $x \in H$, then $xH = H$. What is the interpretation of this statement in terms of the Cayley diagram?

   (b) Prove that if $b \in aH$, then $aH = bH$.

   (c) Show that for any $x \in G$, the map

   $$\varphi\colon H \longrightarrow xH\,, \qquad \varphi\colon h \longmapsto xh$$

   is a bijection. Conclude that all left cosets of H have the same size.

   (d) Conclude that G is partitioned by the left cosets of H, all of which are equal size.

6. A subgroup H of G is *normal* if $xH = Hx$ for all $x \in G$. Prove that if $[G : H] = 2$, then H is a normal subgroup of G. [*Hint*: Use the results of the previous problem.]