Read Chapters 8.1–3 of *Visual Group Theory*, or Chapters 4.2, 5.6, 8.1 of *IBL Abstract Algebra*, and then write up solutions to the following exercises.

1. Do the following steps for each mapping $\phi\colon G \to H$ listed below.

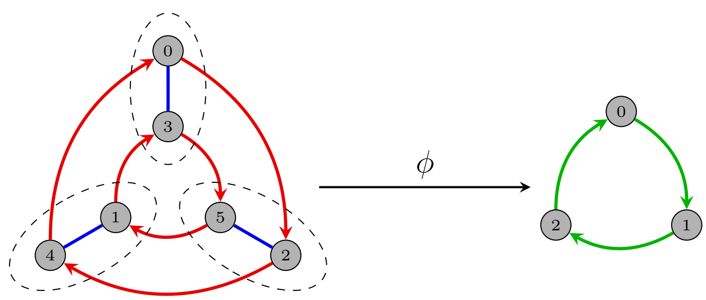
   (i) Find $\ker\phi := \{g \in G \mid \phi(g) = e\}$ and $\operatorname{im}\phi = \phi(G) = \{\phi(g) \mid g \in G\}$.

   (ii) Draw Cayley diagrams of the domain and codomain, and arrange them so one can "visually see" the cosets of $\ker\phi$ in G. Draw dotted lines around these cosets.

   (iii) Is the quotient $G/\ker\phi$ a group? If so, what is it isomorphic to?

   (iv) Is ϕ a homomorphism? If no, explain what is going wrong.

   Here is an example of Step (iii) for the map $\phi\colon \mathbb{Z}_6 \to \mathbb{Z}_3$, defined by $\phi(n) = n \pmod 3$.



   Now, do steps (i)–(iv) for the following maps.

   (a) The map $\phi\colon \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(n) = 4n$.

   (b) The map $\phi\colon D_4 \to \mathbb{Z}_4$ defined by $\phi(r^k f) = k$.

   (c) The map $\phi\colon D_4 \to \mathbb{Z}_4$ defined by $\phi(r^k f) = 2k$.

   (d) The map $\phi\colon \mathbb{Z}_4 \to D_4$ defined by $\phi(k) = r^k$.

   (e) The map $\phi\colon D_4 \to V_4$ defined by $\phi(r) = h$ and $\phi(f) = v$.

2. Prove that $A \times B \cong B \times A$.

3. For each part below, list *all* homomorphisms with the given domain and codomain. It suffices to write down the image of the generators, but you must explain why the map exists.

   (a) Domain $\mathbb{Z}_{15}$ and codomain $\mathbb{Z}_4$.

   (b) Domain $\mathbb{Z}_{412}$ and codomain $\mathbb{Z}_{440}$.

   (c) Domain and codomain both $\mathbb{Z}_4$.

   (d) Domain C_4 and codomain V_4.

   (e) Domain and codomain both V_4.

4. Show that there is no embedding $\phi\colon \mathbb{Z}_n \hookrightarrow \mathbb{Z}$.

5. Let $H \leq G$, and fix $x \in G$. Recall that we showed in class that xHx^{-1} is always a subgroup of G.

   (a) Prove additionally that $xHx^{-1} \cong H$. [*Hint*: Define a mapping from H to xHx^{-1} and prove that it is a homomorphism, one-to-one, and onto.]

   (b) Use Part (a) to show that $|xy| = |yx|$ for any $x, y \in G$.

6. Let $\phi\colon G \to H$ be a homomorphism, and $N \trianglelefteq H$.

   (i) Show that the set $\phi^{-1}(N) := \{g \in G \mid \phi(g) \in N\}$ is a subgroup of G.

   (ii) Show that $\phi^{-1}(N)$ is a *normal* subgroup of G.

   (iii) Show by example that if $M \trianglelefteq G$, then $\phi(M)$ need not be a normal subgroup of H.