Read Chapters 8.4–5 of *Visual Group Theory*, Chapter 9.2 of *IBL Abstract Algebra*, or Chapters 10.3 and 12.1 of *AATA*. Then write up solutions to the following exercises.

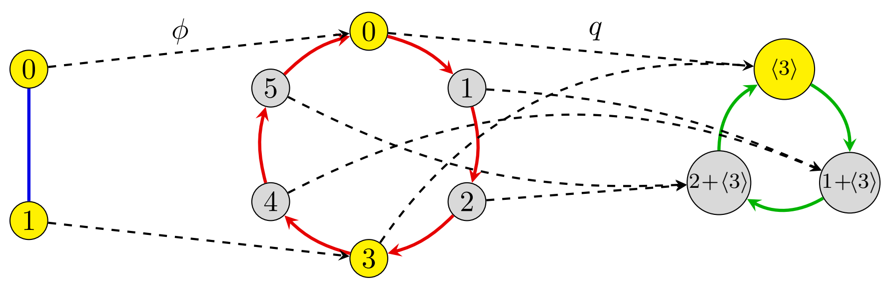
1. Let $(\mathbb{Q},+)$ be the group of rational numbers under addtion, $(\mathbb{Q}^*,\cdot\,)$ the group of non-zero rational numbers under multiplication, and $(\mathbb{Q}^+,\cdot\,)$ the group of positive rational numbers under multiplication.

   (a) Show that $(\mathbb{Q}^*,\cdot\,)\cong(\mathbb{Q}^+,\cdot\,)\times C_2$. [*Hint*: Recall that $C_2=\{e^{0\pi i},e^{\pi i}\}=\{1,-1\}$.]

   (b) Describe the quotient groups $(\mathbb{Q},+)/\langle -1\rangle$ and $(\mathbb{Q}^*,\cdot\,)/\langle -1\rangle$. In particular, what do the elements (cosets) look like?

   (c) Use the Fundamental Homomorphism Theorem to prove that $(\mathbb{Q}^*,\cdot\,)/\langle -1\rangle\cong(\mathbb{Q}^+,\cdot\,)$.

2. For Parts (a)–(d), a group $G$ is given together with a normal subgroup $H$. Illustrate the embedding $\phi\colon H\to G$, and the quotient map $q\colon G\to G/H$, chained together so that $\mathrm{im}(\phi)=\ker(q)$. An example for $G=\mathbb{Z}_6$ and $H=\mathbb{Z}_2$ is shown below:




$H=\mathbb{Z}_2$ $\qquad$ $H\cong\mathrm{im}(\phi)=\langle 3\rangle<G$ $\qquad$ $G/\langle 3\rangle\cong\mathbb{Z}_3$

   (a) $G=\mathbb{Z}_6$, $H=\mathbb{Z}_3$,

   (b) $G=D_3$, $H=C_3$,

   (c) $G=A_4$, $H=V_4$,

   (d) $G=S_n$, $H=A_n$ [don't draw the actual Cayley graphs for this one, just the maps].

   Now, answer each of the following questions about each of your answers to Parts (a)–(d).

   (e) What map $\theta$ into $H$ would satisfy the equation $\mathrm{im}\,\theta=\ker\phi$? Choose one with the smallest possible domain.

   (f) What map $\theta'$ from $G/H$ would satisfy the equation $\mathrm{im}(q)=\ker(\theta')$? Choose one with the smallest possible codomain.

   (g) Add the two maps $\theta$ and $\theta'$ to your illustration.

   (h) The new chain of four homomorphisms is called a *short exact sequence*. It is one way to use homomorphisms to illustrate quotients, and it shows a connection betwen embeddings and quotient maps. Given a normal subgroup $H\trianglelefteq G$, show how to create a short exact sequence involving $G$ and $H$.

3. Let $A \leq B$ and $B \trianglelefteq G$. In this problem, you will prove the *Diamond Isomorphism Theorem*.

   (a) Show that $AB := \{ab : a \in A, b \in B\}$ and $BA := \{ba : a \in A, b \in B\}$ are equal as sets.

   (b) Show that $AB$ is a subgroup of $G$.

   (c) Show that $B \trianglelefteq AB$ and $A \cap B \trianglelefteq A$.

   (d) Show that $A/(A \cap B) \cong AB/B$. [*Hint:* Construct a homomorphism $\phi\colon A \to AB/B$ that has kernel $A \cap B$, then apply the FHT.]

   (e) Draw a diagram, or lattice, of $G$ and its subgroups $AB$, $A$, $B$, and $A \cap B$. Interpret the result in Part (c) in terms of this diagram.

4. For each part below, consider the group $G = \langle A, B \rangle$ generated by the two matrices given. Assume that matrix multiplication is the binary operation, and $i = \sqrt{-1}$. To what common group is $G$ isomorphic? Write down an explicit isomorphism (you only need to define it for the generators), and a group presentation for $G$.

   (a) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

5. In this exercise, you will prove that if $A$ and $B$ are normal subgroups and $AB = G$, then
$$G/(A \cap B) \cong (G/A) \times (G/B).$$

   (a) Consider the following map:
$$\phi\colon AB \longrightarrow (G/A) \times (G/B), \qquad \phi(g) = (gA, gB).$$
   Show that $\phi$ is a homomorphism.

   (b) Show that $\phi$ is surjective. That is, given any $(g_1A, g_2B)$, show that there is some $g = ab \in AB$ such that $\phi(g) = (g_1A, g_2B)$. [*Hint*: Try $g = a_2b_1$.]

   (c) Find $\ker(\phi)$ [you need to prove your answer is correct] and then apply the Fundamental Homomorphism Theorem.

6. For the numbers below, list all abelian groups of that order by writing each one as a product of cyclic groups of prime power order. Then, determine which group it is isomorphic to of the form $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_{i+1} \mid n_i$.

   (a) 16

   (b) 54

   (c) 400

   (d) $p^2q$, where $p$ and $q$ are distinct primes.