Read Chapters 9.2–9.4 of *Visual Group Theory* (VGT), or Chapters 13–14 of *AATA*. Then write up solutions to the following exercises.

1. Let S be the following set of 7 "binary squares":

$$S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

   (a) Consider the (right) action of the group $G = V_4 = \langle v, h\rangle$ on S, where $\phi(v)$ reflects each square vertically, and $\phi(h)$ reflects each square horizontally. Draw an action diagram and compute the stabilizer of each element.

   (b) Consider the (right) action of the group $G = C_4 = \langle r \mid r^4 = e\rangle$ on S, where $\phi(r)$ rotates each square $90°$ clockwise. Draw an action diagram and compute the stabilizer of each element.

   (c) Suppose a group G of order 15 acts on S. Show that there must be a fixed point.

2. Let $G = S_4$ act on itself by conjugation via the homomorphism

$$\phi \colon G \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

   (a) How many orbits are there? Describe them as specifically as you can.

   (b) Find the orbit and the stabilizer of the following elements:

      i. e
      ii. $(1\ 2)$
      iii. $(1\ 2\ 3)$
      iv. $(1\ 2\ 3\ 4)$
      v. $(1\ 2)\,(3\ 4)$

3. A *p-group* is a group of order p^k for some integer k. Recall that the *center* of a group G is the set of all elements that commute with everything:

$$Z(G) = \{z \in G \mid gz = zg,\ \forall g \in G\} = \{z \in G \mid g^{-1}zg = z,\ \forall g \in G\}\,.$$

   Finally, a group G is *simple* if its only normal subgroups are G and $\langle e\rangle$.

   (a) Let G act on itself by conjugation via the homomorphism

$$\phi \colon G \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

   Show that $\text{Fix}(\phi) = Z(G)$.

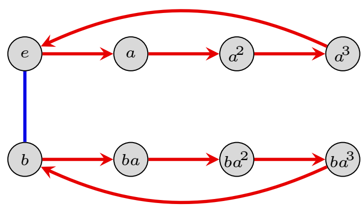
   (b) Show that if G is a p-group, then $|Z(G)| > 1$. [*Hint*: Revisit the Class Equation.]

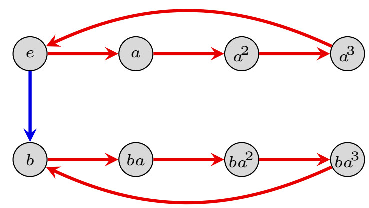
   (c) Use the result of the previous part to classify all simple p-groups.

4. Let G be an unknown group of order 8. By the First Sylow Theorem, G must contain a subgroup H of order 4.

   (a) If all subgroups of G of order 4 are isomorphic to V_4, then what group must G be? Completely justify your answer.

   (b) Next, suppose that G has a subgroup $H \cong C_4$. Then G has a Cayley diagram like one of the following:





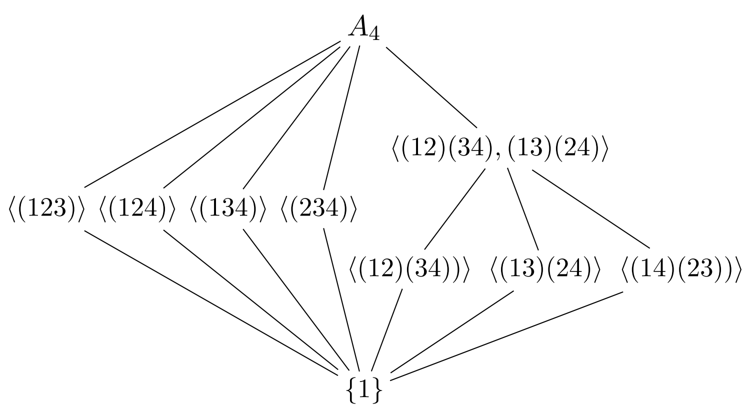
   Find all possibilities for finishing the Cayley diagram.

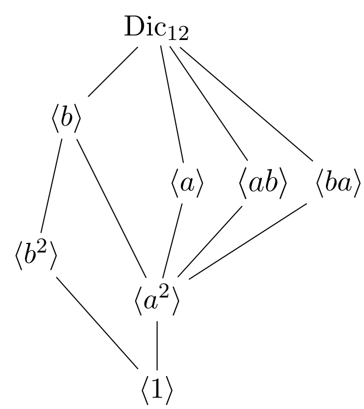
   (c) Label each completed Cayley diagram by isomorphism type. Justify your answer.

   (d) Make a complete list of all groups of order 8, up to isomorphism.

5. In this problem, we will find the Sylow subgroups of all 3 nonabelian groups of order 12.

   (a) Find all Sylow 2-subgroups and Sylow 3-subgroups of the groups whose subgroup lattices are shown below. Determine which are normal





   (b) Find all Sylow subgroups of $D_6 = \langle r, f \mid r^6 = f^2 = e, rfr = f\rangle$, and determine which are normal.

6. Recall that a group G is called *simple* if its only normal subgroups are G and $\{e\}$. Use group actions and/or the Sylow theorems to show the following.

   (a) There is no simple group of order $45 = 3^2 \cdot 5$.

   (b) There is no simple group of order pq, where $p < q$ and are both prime.

   (c) There is no simple group of order $56 = 2^3 \cdot 7$.

   (d) There is no simple group of order $108 = 2^2 \cdot 3^3$.

   (e) If G has a subgroup H with $[G : H] = p$, the smallest prime dividing $|G|$, then $H \trianglelefteq G$, and hence G cannot be simple. [*Hint*: Let G act on the right cosets of H.]