1. Let $p \in \mathbb{N}$ be a fixed prime. For each of the three ideals $I = (p)$, (x), and (x, p) in the ring $R = \mathbb{Z}[x]$, do the following steps:
   (i) Describe the elements of the ideal formally, as $I = \{ \quad : \quad \}$.
   (ii) Characterize the polynomials in I in plain English.
   (iii) Determine whether I is maximal and/or prime.
   (iv) Describe the quotient ring R/I.

   Then, repeat the above steps for these ideals but in the ring $\mathbb{Q}[x]$.

2. Let R be a commutative ring with 1.
   (a) Prove that R is an integral domain if and only if 0 is a prime ideal.
   (b) Prove that an ideal $P \subseteq R$ is prime if and only if R/P is an integral domain.
   (c) Show that every maximal ideal is prime.
   (d) Find the group of units $U(R)$ and the maximal ideal(s) of the ring

   $$R = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ \gcd(a,b) = 1,\ p \nmid b \right\},$$

   where p is a fixed prime number.

3. Let R be a principal ideal domain (PID). A *common multiple* of $a, b \in R^*$ is an element m such that $a \mid m$ and $b \mid m$. Moreover, m is a *least common multiple* (LCM) if $m \mid n$ for any other common multiple n of a and b.
   (a) Prove that any $a, b \in R^*$ have an LCM.
   (b) Prove that an LCM of a and b is unique up to multiplication of associates, and can be characterized as a generator of the (principal) ideal $I := (a) \cap (b)$.

4. For any $x = r + s\sqrt{m} \in \mathbb{Q}(\sqrt{m})$, define the *norm* of x to be $N(x) = r^2 - ms^2$.
   (a) Show that $N(xy) = N(x)N(y)$.
   (b) Show that $N(x) \in \mathbb{Z}$ if $x \in R_m$.
   (c) Show that $u \in U(R_m)$ if and only if $|N(u)| = 1$.
   (d) Show that $U(R_{-1}) = \{\pm 1, \pm i\}$, $U(R_{-3}) = \{\pm 1, \pm(1 \pm \sqrt{3})/2\}$, and $U(R_m) = \{\pm 1\}$ for all other negative square-free $m \in \mathbb{Z}$.