Lecture 3.2: Cosets

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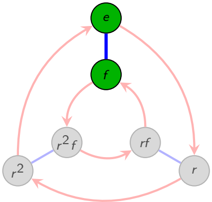
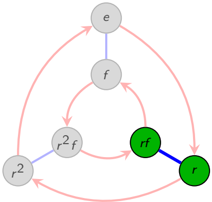
Math 4120, Modern Algebra

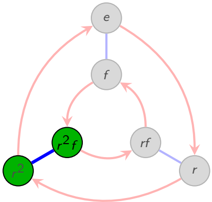
Overview

The regularity property of Cayley diagrams implies that identical copies of the fragment of the diagram that correspond to a subgroup appear throughout the rest of the diagram.

For example, the following figures highlight the repeated copies of $\langle f \rangle = \{e, f\}$ in D_3:







However, only one of these copies is actually a group! Since the other two copies do *not* contain the identity, they cannot be groups.

Key concept

The elements that form these repeated copies of the subgroup fragment in the Cayley diagram are called cosets.

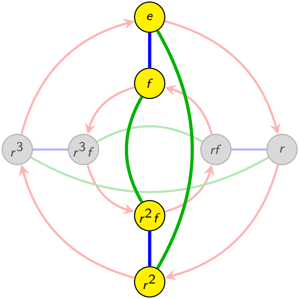
An example: D_4

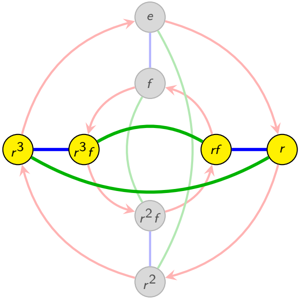
Let's find all of the cosets of the subgroup $H = \langle f, r^2 \rangle = \{e, f, r^2, r^2 f\}$ of D_4.

If we use r^2 as a generator in the Cayley diagram of D_4, then it will be easier to "see" the cosets.

Note that $D_4 = \langle r, f \rangle = \langle r, f, r^2 \rangle$. The cosets of $H = \langle f, r^2 \rangle$ are:

$$H = \langle f, r^2 \rangle = \underbrace{\{e, f, r^2, r^2 f\}}_{\text{original}}, \qquad rH = r\langle f, r^2 \rangle = \underbrace{\{r, r^3, rf, r^3 f\}}_{\text{copy}}.$$





More on cosets

Definition

If H is a subgroup of G, then a (left) coset is a set

$$aH = \{ah : h \in H\},$$

where $a \in G$ is some fixed element. The distingusihed element (in this case, a) that we choose to use to name the coset is called the representative.

Remark

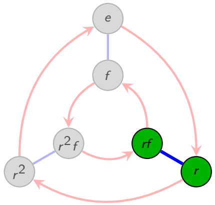
In a Cayley diagram, the (left) coset aH can be found as follows: start from node a and follow *all* paths in H.

For example, let $H = \langle f \rangle$ in D_3. The coset $\{r, rf\}$ of H is the set

$$rH = r\langle f \rangle = r\{e, f\} = \{r, rf\}.$$

Alternatively, we could have written $(rf)H$ to denote the same coset, because

$$rfH = rf\{e, f\} = \{rf, rf^2\} = \{rf, r\}.$$



More on cosets

The following results should be "visually clear" from the Cayley diagrams and the regularity property. Formal algebraic proofs that are not done here will be assigned as homework.

Proposition

For any subgroup $H \leq G$, the union of the (left) cosets of H is the whole group G.

Proof

The element $g \in G$ lies in the coset gH, because $g = ge \in gH = \{gh \mid h \in H\}$. $\square$

Proposition

Each (left) coset can have multiple representatives. Specifically, if $b \in aH$, then $aH = bH$. $\square$

Proposition

All (left) cosets of $H \leq G$ have the same size. $\square$

More on cosets

Proposition

For any subgroup $H \leq G$, the (left) cosets of H partition the group G.

Proof

We know that the element $g \in G$ lies in a (left) coset of H, namely gH. Uniqueness follows because if $g \in kH$, then $gH = kH$. $\square$

Subgroups also have right cosets:

$$Ha = \{ha \colon h \in H\}.$$

For example, the right cosets of $H = \langle f \rangle$ in D_3 are

$$Hr = \langle f \rangle r = \{e, f\}r = \{r, fr\} = \{r, r^2 f\}$$

(recall that $fr = r^2 f$) and

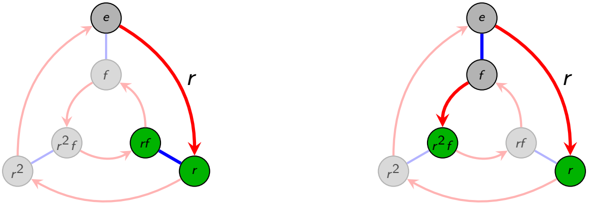
$$\langle f \rangle r^2 = \{e, f\}r^2 = \{r^2, fr^2\} = \{r^2, rf\}.$$

In this example, the left cosets for $\langle f \rangle$ are different than the right cosets. Thus, they must look different in the Cayley diagram.

Left vs. right cosets

The left diagram below shows the left coset $r\langle f \rangle$ in D_3: the nodes that f arrows can reach after the path to r has been followed.

The right diagram shows the right coset $\langle f \rangle r$ in D_3: the nodes that r arrows can reach from the elements in $\langle f \rangle$.



Thus, left cosets look like copies of the subgroup, while the elements of right cosets are usually scattered, because we adopted the convention that arrows in a Cayley diagram represent right multiplication.

Key point

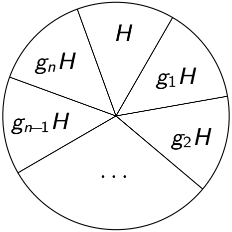
Left and right cosets are generally different.

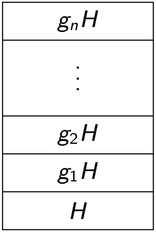
Left vs. right cosets

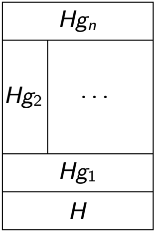
For any subgroup $H \leq G$, we can think of G as the union of non-overlapping and equal size copies of *any* subgroup, namely that subgroup's left cosets.

Though the right cosets also partition G, the corresponding partitions could be different!

Here are a few visualizations of this idea:







Definition

If $H < G$, then the **index** of H in G, written $[G : H]$, is the number of distinct left (or equivalently, right) cosets of H in G.

Left vs. right cosets

The left and right cosets of the subgroup $H = \langle f \rangle \leq D_3$ are *different*:

r^2H	r^2f	r^2
rH	r	rf
H	e	f

Hr	r^2f	r^2	Hr^2
	r	rf	
H	e	f	

The left and right cosets of the subgroup $N = \langle r \rangle \leq D_3$ are *the same*:

fN	f	rf	r^2f
N	e	r	r^2

Nf	f	rf	r^2f
N	e	r	r^2

Proposition

If $H \leq G$ has index $[G : H] = 2$, then the left and right cosets of H are the same.

Cosets of abelian groups

Recall that in some abelian groups, we use the symbol $+$ for the binary operation.

In this case, left cosets have the form $a+H$ (instead of aH).

For example, let $G=(\mathbb{Z},+)$, and consider the subgroup $H=4\mathbb{Z}=\{4k \mid k\in\mathbb{Z}\}$ consisting of multiples of 4.

The left cosets of H are

$$\begin{aligned} H &= \{\dots,-12,-8,-4,0,4,8,12,\dots\} \\ 1+H &= \{\dots,-11,-7,-3,1,5,9,13,\dots\} \\ 2+H &= \{\dots,-10,-6,-2,2,6,10,14,\dots\} \\ 3+H &= \{\dots,-9,-5,-1,3,7,11,15,\dots\}. \end{aligned}$$

Notice that these are the same the the right cosets of H:

$$H\,, \qquad H+1\,, \qquad H+2\,, \qquad H+3\,.$$

Do you see why the left and right cosets of an abelian group will *always* be the same?

Also, note why it would be incorrect to write $3H$ for the coset $3+H$. In fact, $3H$ would probably be interpreted to be the subgroup $12\mathbb{Z}$.

A theorem of Joseph Lagrange

We will finish with one of our first major theorems, named after the prolific 18th century Italian/French mathematician Joseph Lagrange.

Lagrange's Theorem

Assume G is finite. If $H < G$, then $|H|$ divides $|G|$.

Proof

Suppose there are n left cosets of the subgroup H. Since they are all the same size, and they partition G, we must have

$$|G| = \underbrace{|H| + \cdots + |H|}_{n \text{ copies}} = n\,|H|.$$

Therefore, $|H|$ divides $|G|$. □

Corollary

If $|G| < \infty$ and $H \leq G$, then

$$[G : H] = \frac{|G|}{|H|}.$$