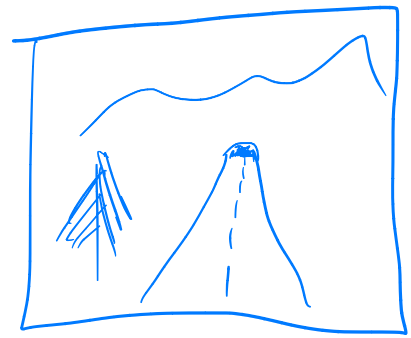
Part 1: What is infinity?

Math 1060
Fall 2020
8/24 – 8/26

Mon 8/24

$\boxed{\text{Infinity}}$



What do we mean by "infinity"?

Numbers? Lines? Space? Something else?

How does infinity arise in art & architecture?

Can we do math with infinity?

Intuitive: $\frac{1}{0} = \infty$, $\frac{-1}{0} = -\infty$, $\frac{1}{\infty} = 0$, $\infty + \infty = \infty$

less clear: $\frac{0}{0} = ?$ $\frac{\infty}{\infty} = ?$ $\infty - \infty = ?$

Take away message: Infinity behaves weird.

<u>Motivating example</u>

2 farmers plant 1 seed every day

Farmer 1: 1 2 3 ~~4~~ 5 6 7 ~~8~~ 9 10 11 ~~12~~ 13 ...

Farmer 2: ~~1~~ ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12 13 ...

A bird eats a seed every 4 days.

How many seeds are left "at the end of time?"

<u>Farmer 1</u>: "$\infty - \infty$" $= \infty$ <u>Farmer 2</u> $\infty - \infty = 0$

Question What is infinite?

Are all infinities the $\underbrace{\text{same "size"}}$?

↑ what does this even mean?

$\infty + \infty = \infty$

$2 \cdot \infty = \infty$

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$ "natural numbers"

$2\mathbb{N} = \{2, 4, 6, 8, 10, 12, \dots\}$ "even #'s"

$\mathbb{N} > 2\mathbb{N}$?

OR $\mathbb{N} = 2\mathbb{N}$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ "integers"

countably infinite

1	2	3	4	5	6	7	8	9	10	11	...
•	•	•	•	•	•	•	•	•	•	•	...
2	4	6	8	10	12	14	16	18	20	22	...
0	1	-1	2	-2	3	-3	4	-4	5	-5	...

$\mathbb{Q} = \{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, \gcd(a,b) = 1\}$ "rationals"

$\mathbb{R} = \{\text{all real numbers}\}$ "reals"

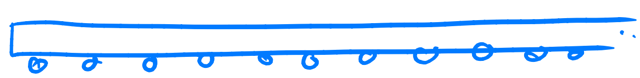
uncountable

Note: $2\mathbb{Z} = \mathbb{N} = \mathbb{Z} = \mathbb{Q} \leq \mathbb{R}$

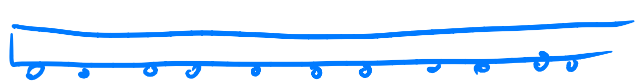
Hilbert's hotel

1	2	3	4	5	6	7	8	9	10	...

Infinite bus



$\mathbb{N}$

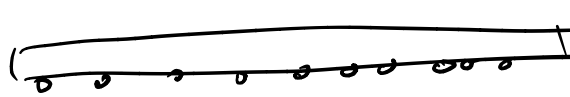


$\mathbb{Z}$

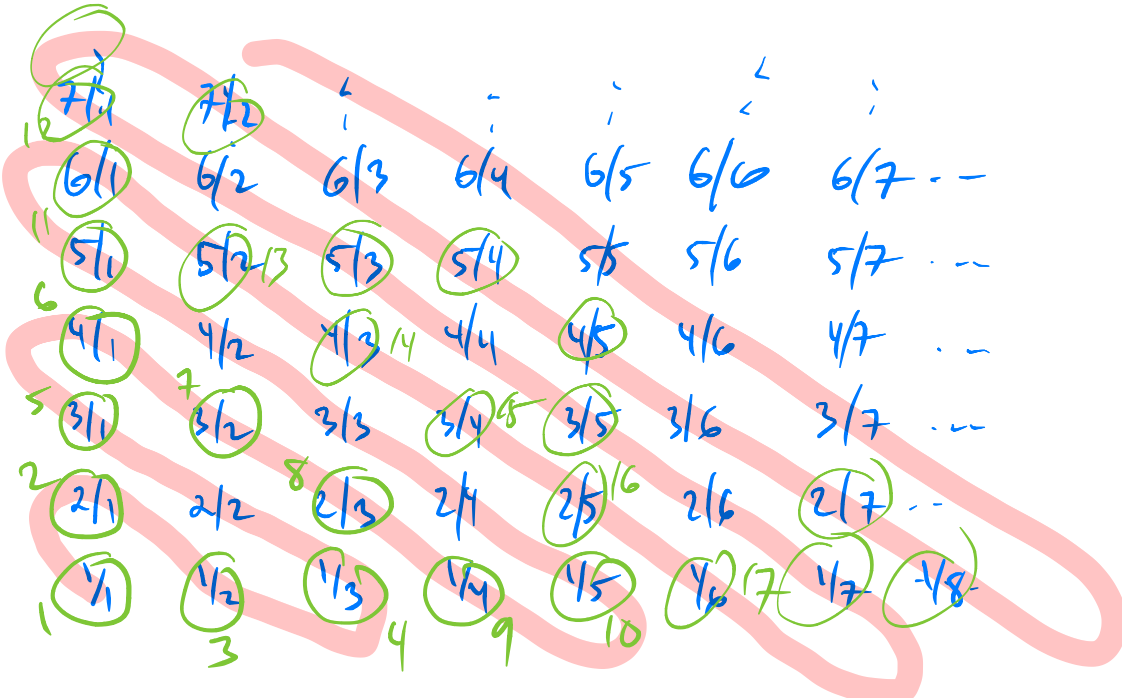
$-5, -23$

$\mathbb{Q}^+$

$\frac{8}{25}, \frac{1}{2}, \frac{2}{31}, \frac{101}{408}, \dots$



$\mathbb{R}^+$



Claim: $|\mathbb{N}| < |\mathbb{R}|$ ★

Suppose that $|\mathbb{N}| = |\mathbb{R}|$ i.e, that everybody gets a room.

Room 1	0.123480263180...
Room 2	3.00120128 9912...
Room 3	16.897612223618...
Room 4	4.018762349166...
Room 5	2.754216341692...
Room 6	8.088264026128...
Room 7	3.141592585213...
⋮	⋮

n^{th} digit of n^{th} #

$x = 0.2188256...$ homeless

Contradiction to (★)

$|\mathbb{R}|$ is a larger infinity than $|\mathbb{Z}|$!

Georg Cantor 1891

$|\mathbb{Q}| < |\mathbb{R}|$

"continuum hypothesis"

Question: Is there an infinite b/w these?

1940: Kurt Gödel "incompleteness theorem"

1963: Paul Cohen continuum hypothesis is undecidable

Goldbach conjecture: Every even # >2 is the sum of 2 primes.

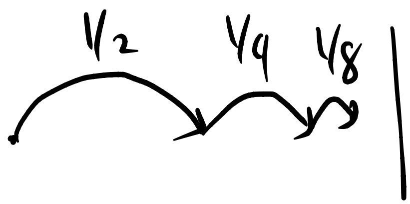
$4 = 2+2$, $6 = 3+3$, $8 = 3+5$, $10 = 3+7 = 5+5$

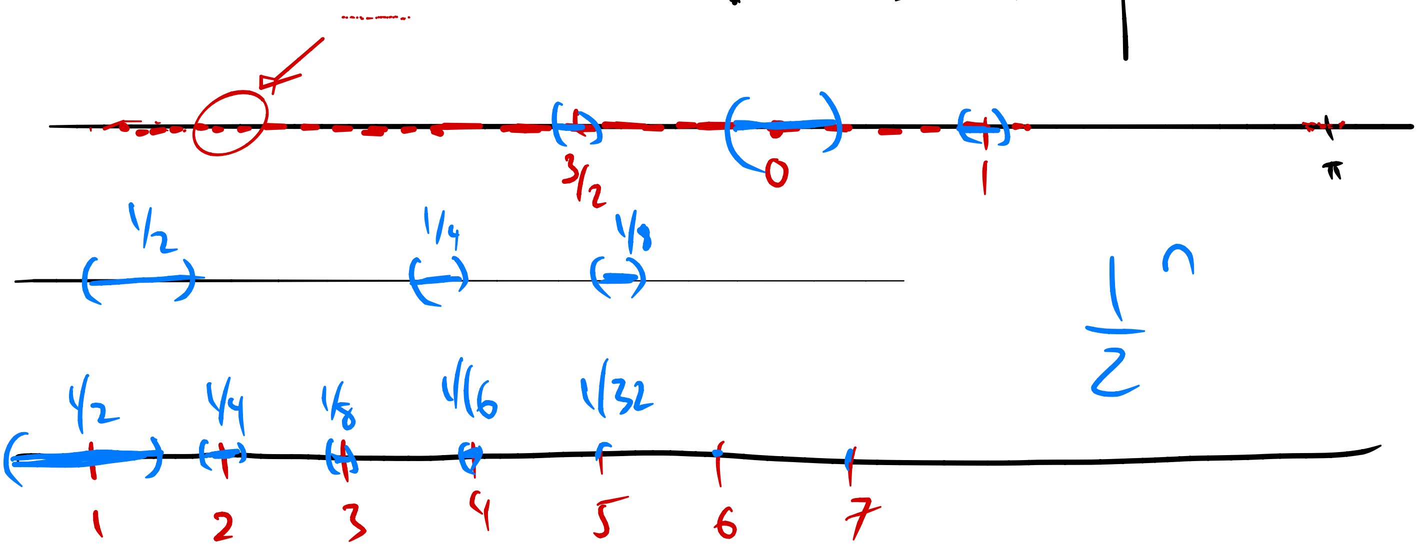
unproven

undecidable??

Fun Facts about $|\mathbb{N}| = |\mathbb{Q}| < |\mathbb{R}|$

① We can "cover" the rational #'s w/ intervals of total length 1





$$\frac{1}{2}^n$$

$$1/2 + 1/4 + 1/8 + 1/16 + \dots = 1$$

② # of binary strings is countable

0011100010101011

# of possible computer programs is countable. $< |\mathbb{R}|$

$\Longrightarrow$ there are uncomputable real #'s.