Part 2: Functions and limits

Math 1060

Fall 2020

8/27 – 9/2

Review of functions

Thurs 8/27

Domain: All possible inputs

Range: All possible outputs

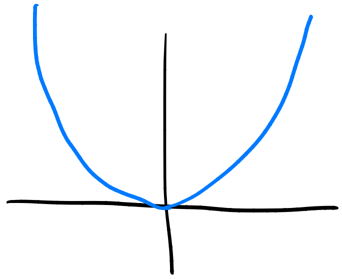
"in"

open interval $x \in (a, b)$ $a < x < b$

closed interval $x \in [a, b]$ $a \le x \le b$

Examples:

① $f(x) = x^2$



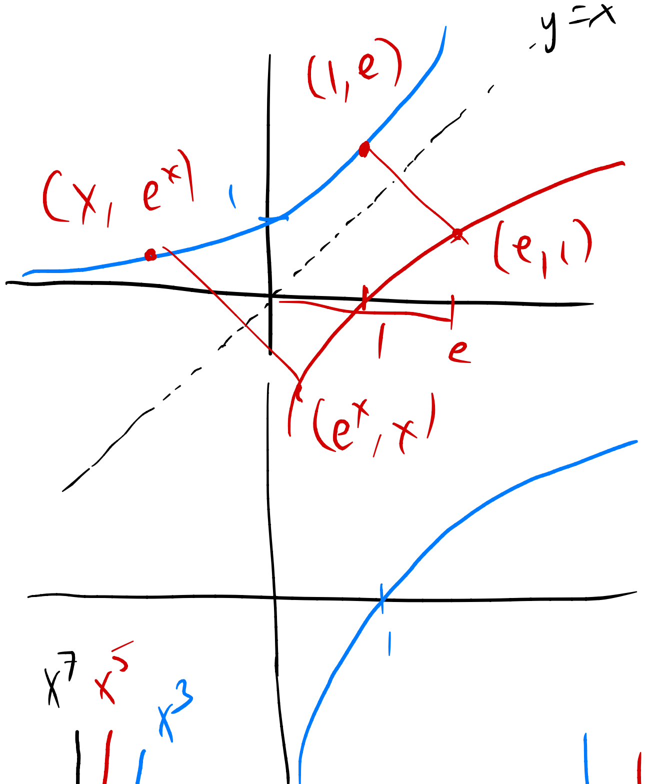
D: $(-\infty, \infty)$ or $\mathbb{R}$

or $-\infty < x < \infty$

R: $[0, \infty)$

or $0 \le x < \infty$ or $f(x) \ge 0$

② $f(x) = e^x$



D: $(-\infty, \infty)$

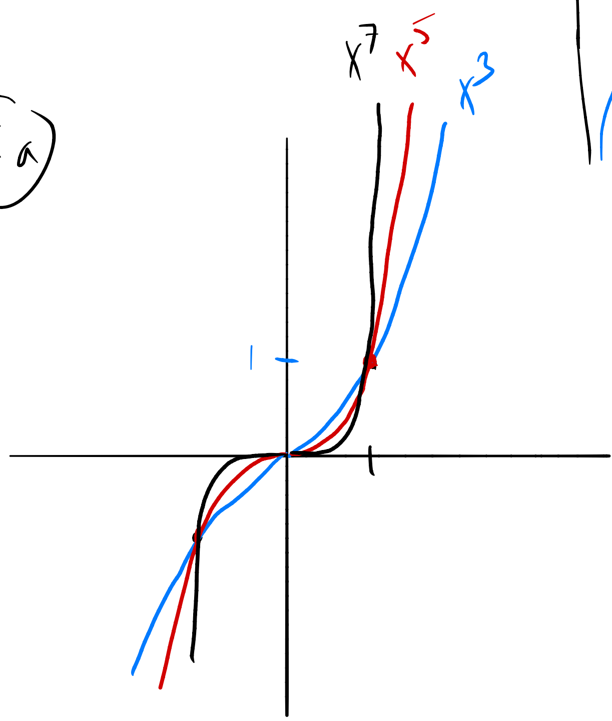
R: $(0, \infty)$ or $f(x) > 0$

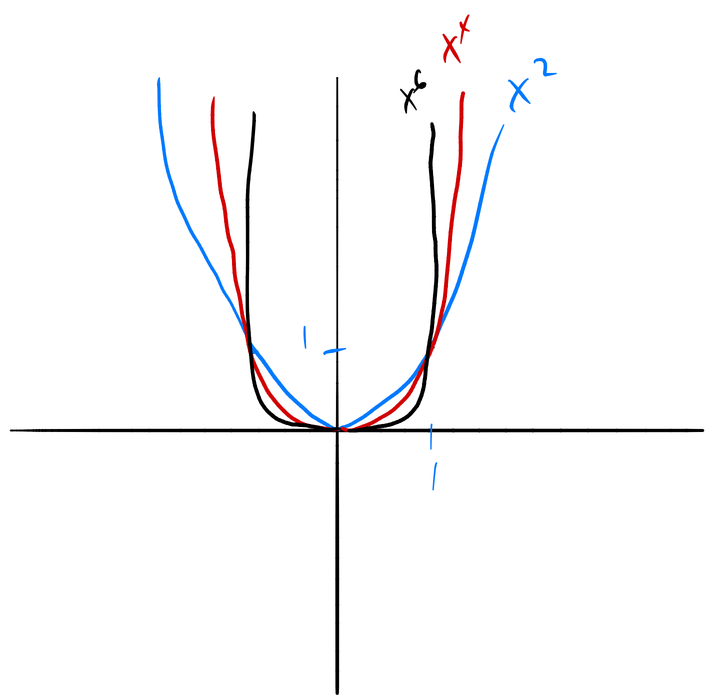
③ $f(x) = \ln x$

D: $(0, \infty)$

R: $(-\infty\ \infty)$

1a

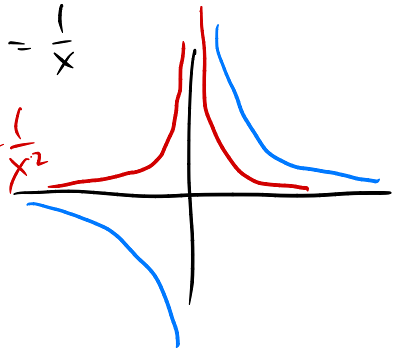




Fri 8/28

④ $f(x) = \frac{1}{x}$

$f(x) = \frac{1}{x^2}$



D: $x \neq 0$; $(-\infty, 0) \cup (0, \infty)$;

$x < 0$ or $x > 0$

R: $f(x) \neq 0$; $(-\infty, 0) \cup (0, \infty)$;

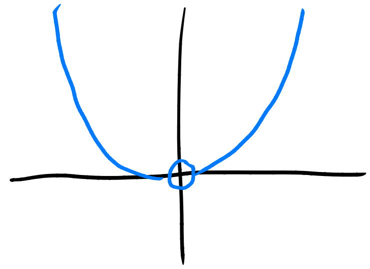
$f(x) < 0$ or $f(x) > 0$.

R: $f(x) > 0$

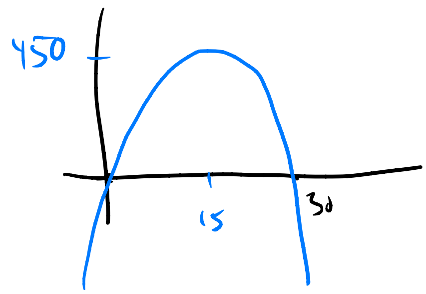
⑤ $f(x) = \frac{x^3}{x} = x^2$ if $x \neq 0$.

D: $x \neq 0$

R: $f(x) > 0$



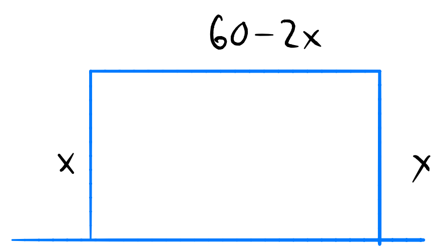
⑥ $f(x) = x(60 - 2x)$



D: $(-\infty, \infty)$

R: $f(x) \leq 450$

⑦ Let $A(x)$ = area of rectangular pen built w/ 60 ft of fence, one side an existing brick wall



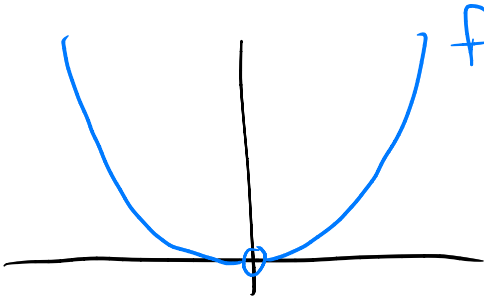
D: $0 < x < 30$

R: $0 < A(x) < 450$

$\boxed{\text{Limits}}$ Math 4530

Def: (informal) $\lim_{x \to a} f(x)$ is what $f(a)$ "should be"

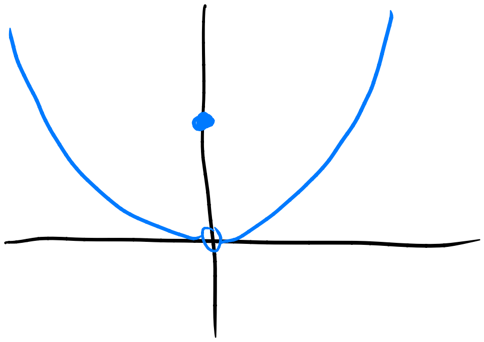
Ex:



$$f(x) = \begin{cases} x^2 & x \neq 0 \\ ? & x = 0 \end{cases}$$

$\lim_{x \to 0} f(x) = 0$

Ex 2:



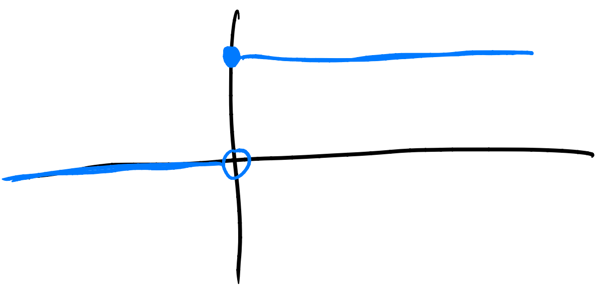
$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad D: \mathbb{R}$$

$$\underbrace{\lim_{x\to 0} f(x)}\ = 0 \neq f(x)$$

f is not continuous.

We can define the

- left-hand limit
- right-hand limit



$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= 0 \\ \lim_{x\to 0^+} f(x) &= 1 \end{aligned}\right\} \Rightarrow \lim_{x\to 0} f(x) \text{ DNE}$$

does not exist.

Note: $f(0) = 1$

Def: The limit of $f(x)$ at $x=a$ exists when the left & right hand limits of $f(x)$ at $x=a$ exist. We write

$$\lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

Def: The function $f(x)$ is continuous at $x=a$ if

$$\boxed{\lim_{x\to a} f(x) = f(a).}$$

Limits at infinity: We can also define

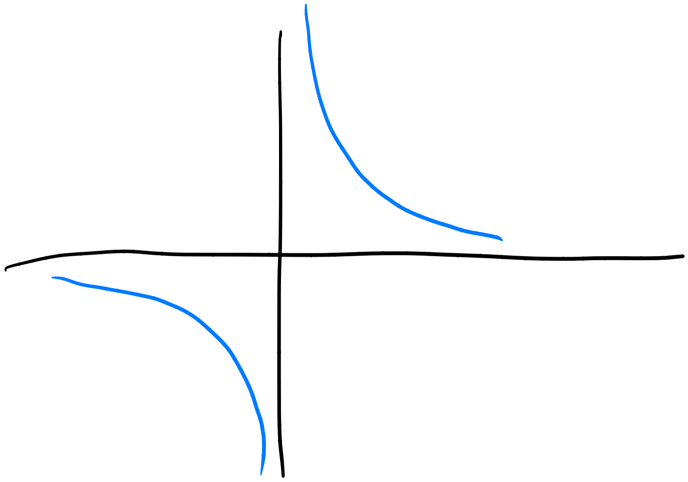
$$\lim_{x\to\infty} f(x) \quad \text{or} \quad \lim_{x\to -\infty} f(x),$$

though they may or may not exist.

Ex 5: $f(x) = \frac{1}{x}$

$\lim_{x \to 0^-} f(x) = -\infty$ (DNE)

$\lim_{x \to 0^+} f(x) = \infty$ (DNE)

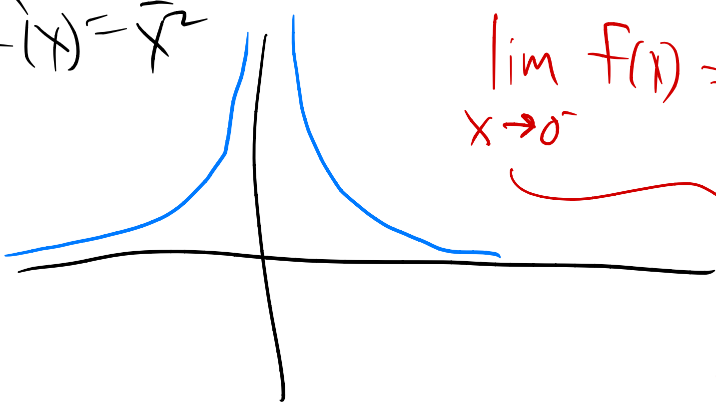


$\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = 0$

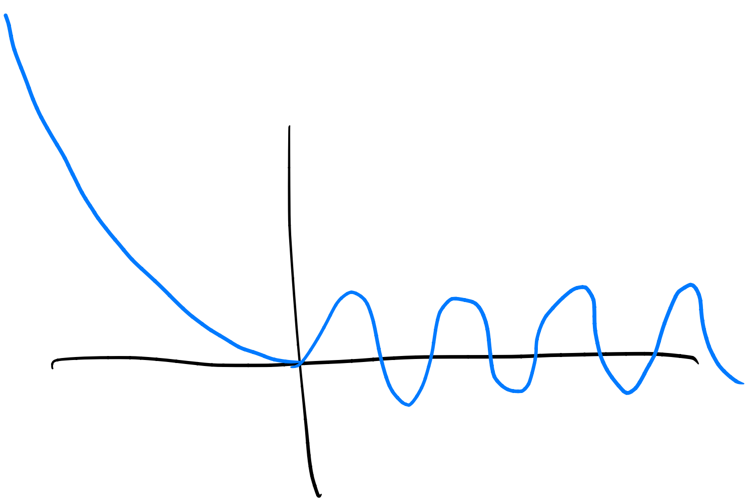
Non-examples:

$f(x) = \frac{1}{x^2}$



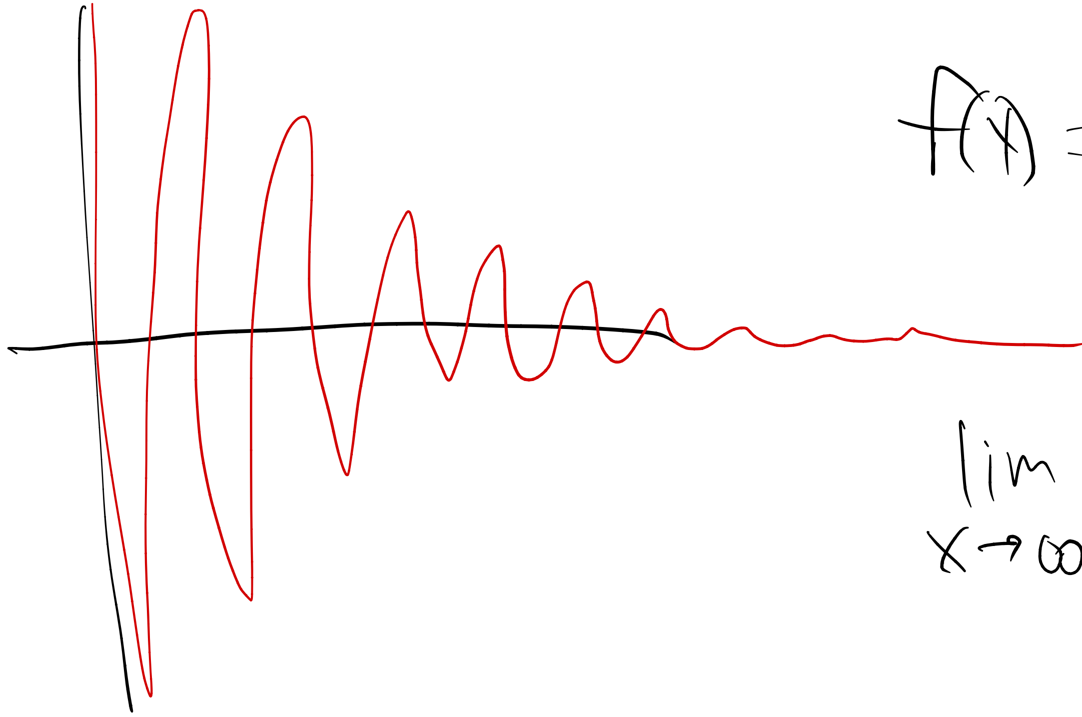
$$\underbrace{\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \infty}$$

technically DNE; otherwise $f(x)$ would be continuous.



$\lim_{x \to -\infty} f(x) = \infty$ (DNE)

$\lim_{x \to \infty} f(x)$ DNE



$$f(x) = \frac{1}{x}\sin x$$

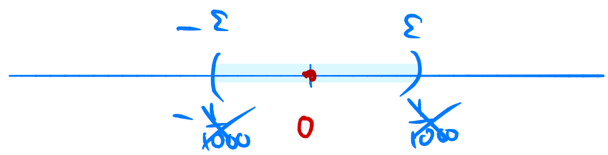
$$\lim_{x\to\infty} f(x) = 0$$

Mon 8/31

Sequences can have limits.

Ex 6: Let $a_n = \frac{1}{n}$. Defines $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots$

We write $\lim_{n\to\infty} a_n = 0$



$\varepsilon = \frac{1}{1000}$

For fun: technical def'n in this example:

For all $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that if $n > m$ then $|a_n - 0| < \varepsilon$.

$\uparrow$ limit

Other examples

$1, 2, 3, 4, 5, 6, 7, 8, \ldots$ limit DNE

$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots$ " "

$1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots$

Question: How to find $\lim_{x\to a} f(a)$?

- First, plug in $x=a$. (this may not work...)

ex: $f(x) = \dfrac{x^2+2}{x+3}$ $\quad \lim_{x\to 0} f(x) = \dfrac{0^2+2}{0+3} = \dfrac{2}{3}$

Note: $\dfrac{\lim_{x\to 0} x^2+2}{\lim_{x\to 0} x+3}$

- If that fails, graph $f(x)$, try to determine $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$

Limits generally behave "like you'd expect"

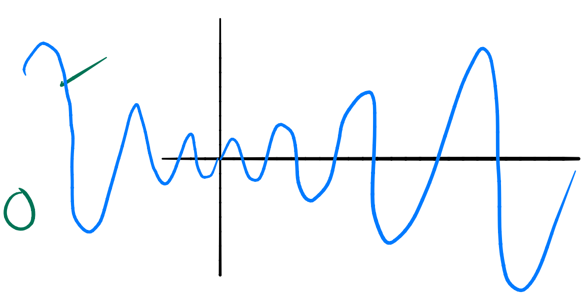
- $\lim_{x\to a} \big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$; if these exist
- $\lim_{x\to a} f(x)\,g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Ex 7: $\lim_{x\to 0} \dfrac{2x^2+3x}{x} = \lim_{x\to 0} \dfrac{2x^2}{x} + \lim_{x\to 0} \dfrac{3x}{x} = \lim_{x\to 0} \dfrac{2x}{1} + \lim_{x\to 0} \dfrac{3}{1}$

Plug in $x=0$: $\dfrac{2\cdot 0+3\cdot 0}{0} = \dfrac{0}{0}$ DNE $\quad = 0 + 3 = 3$

Ex 8: $\lim_{x\to 0} x \sin x$ $\quad$ Plug in $x=0$: $0 \sin 0 = 0$.

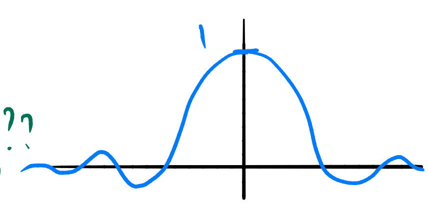
alternatively $\longrightarrow$ $\lim_{x\to 0} x \cdot \lim_{x\to 0} \sin x = 0 \cdot 0 = 0$



Ex 9: $\lim_{x\to 0} \dfrac{\sin x}{x}$ $\quad$ Plug in $x=0$: $\dfrac{\sin 0}{0} = \dfrac{0}{0}$ DNE

$\hookrightarrow \lim_{x\to 0} \dfrac{1}{x} \cdot \lim_{x\to 0} \sin x = \infty \cdot 0 = ???$

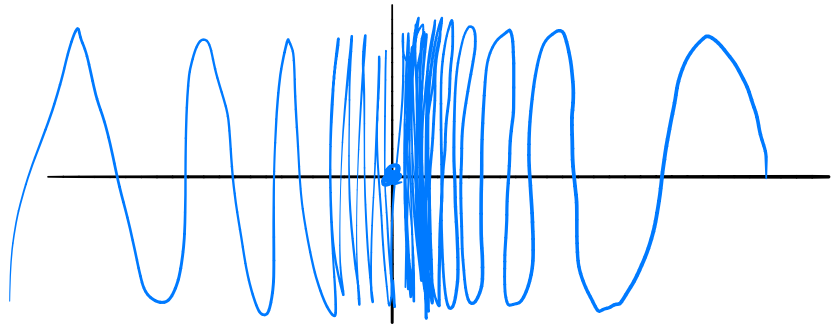
Ans (by graphing): $\boxed{1}$



Other thing can get weird

Ex: $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

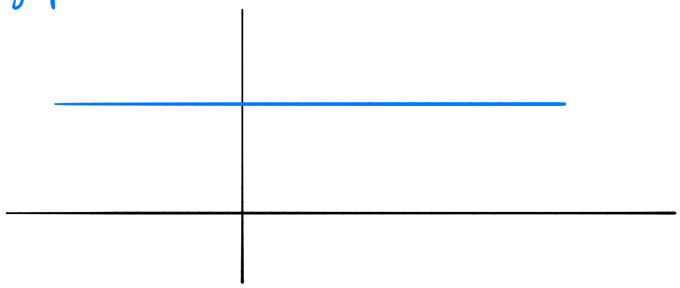
DNE



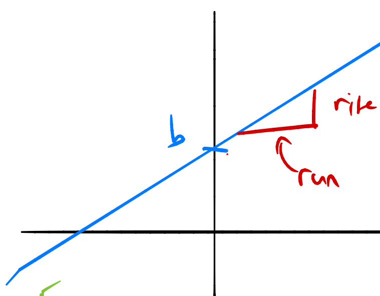
Wed 9/2

Basic functions & how to graph them

Constant: $f(x) = a$



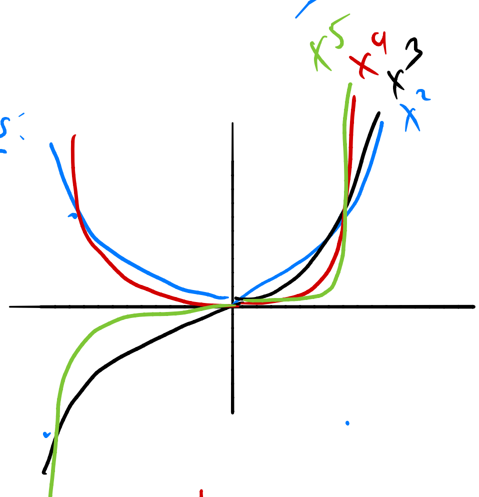
Linear $f(x) = mx + b$



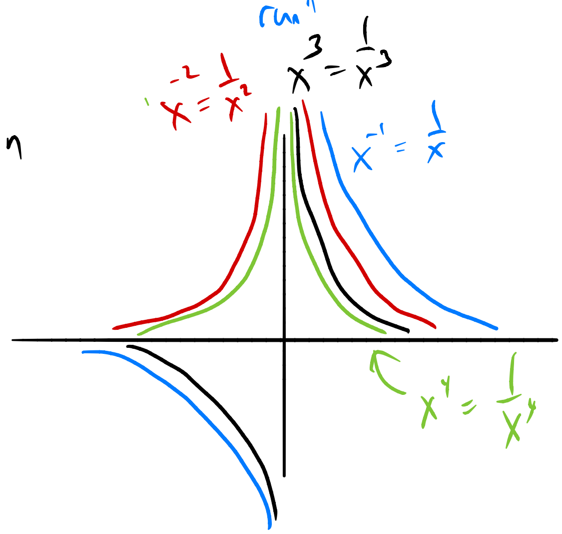
b = y-intercept

m = slope ← $\frac{\text{"rise"}}{\text{"run"}}$

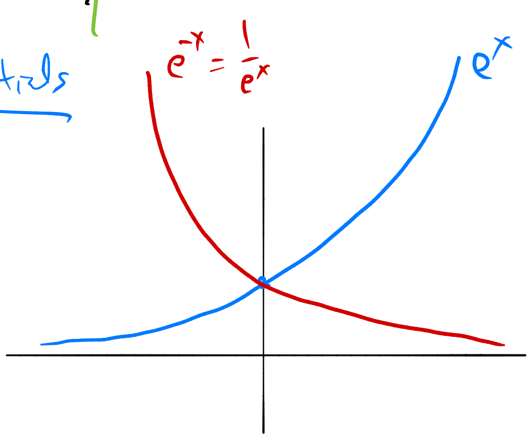
Polynomials:



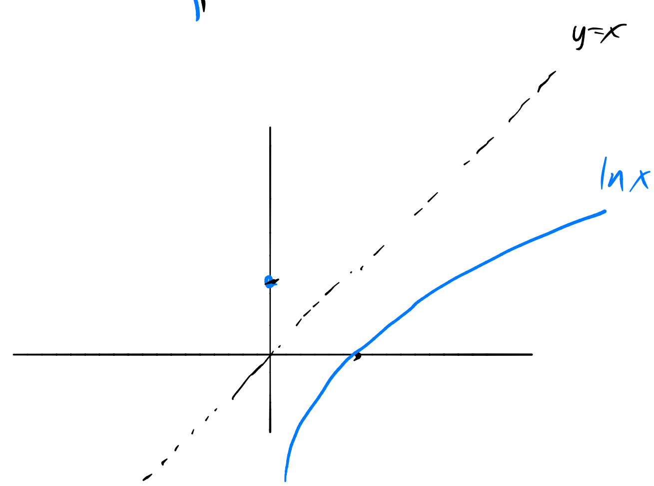
x^{-n}



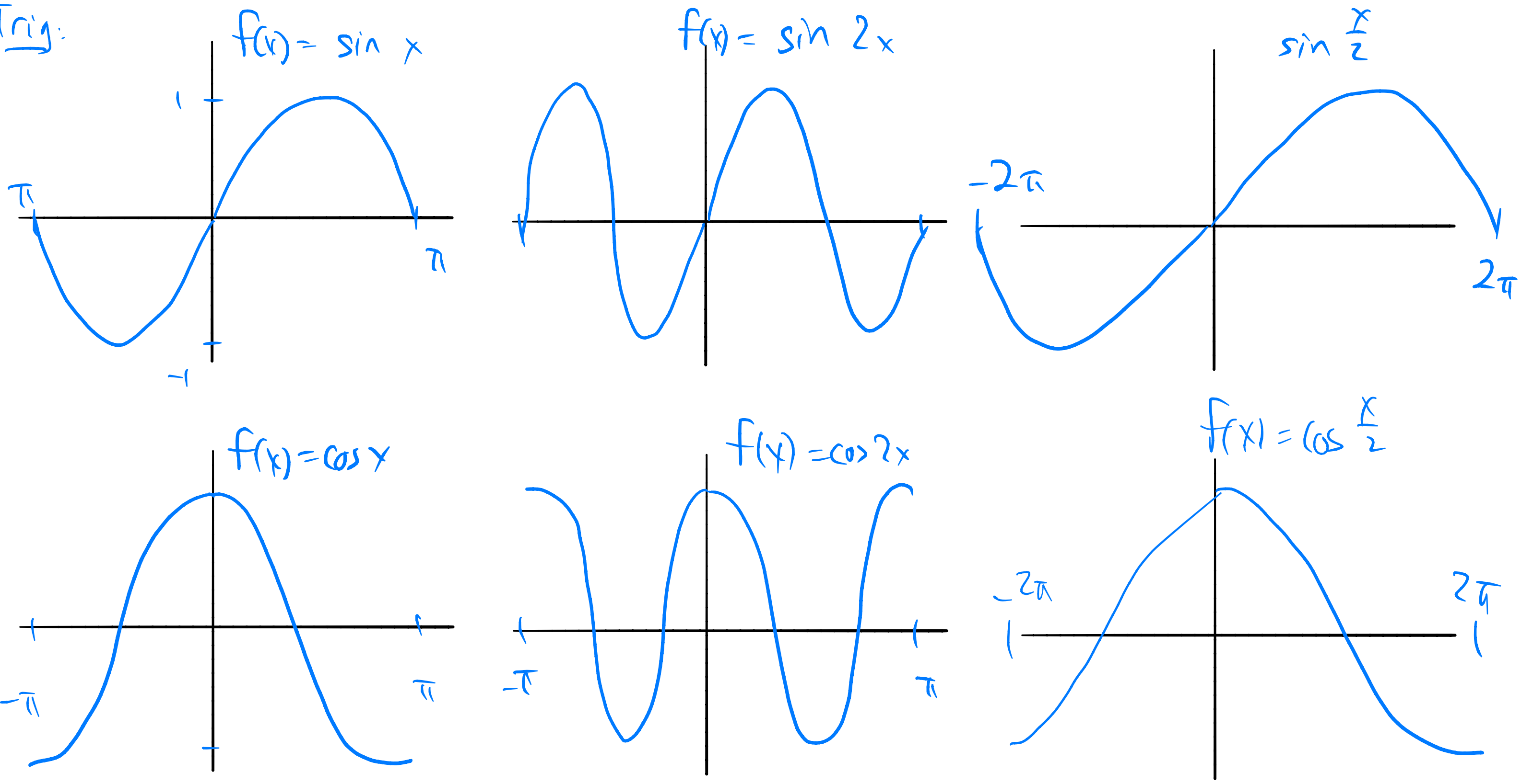
Exponentials



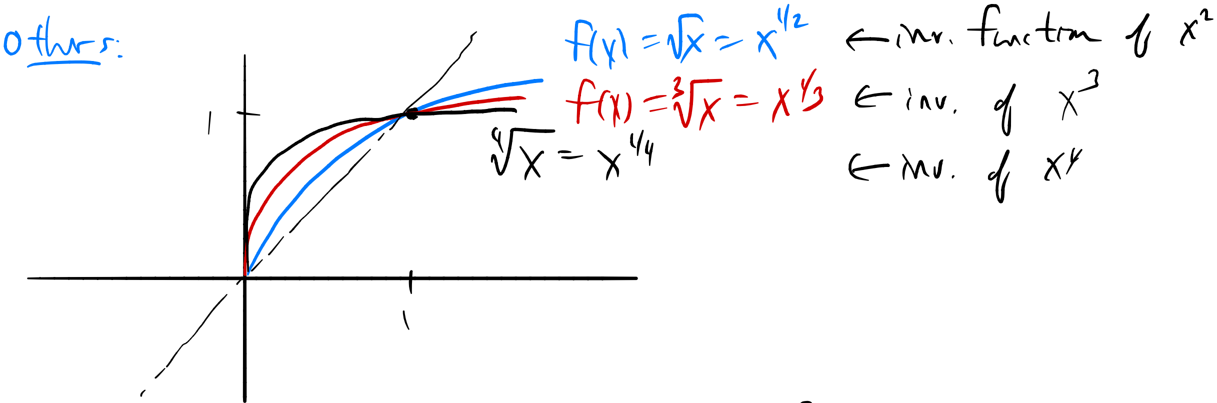
$\ln x$

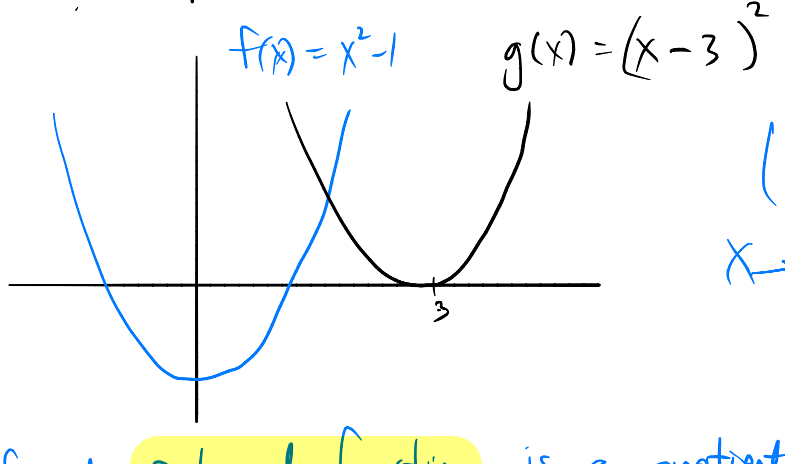


Trig:



Others:





$$\lim_{x\to\infty} \frac{x^2-3}{x^3+4} \approx \frac{100^2-3}{100^3+4} \approx \frac{100^2}{100^3} = \frac{1}{100} \approx 0$$

Def: A rational function is a quotient of two polynomials

Limits of rational functions (what to do, to find $\lim_{x\to a} f(x)$)

1. Plug in $x=a$.
2. Factor top & bottom $\quad \lim_{x\to 1} \frac{x-1}{x^2-6x+5} \overset{\frac{0}{0}\text{ bad}}{\longrightarrow} = \lim_{x\to 1} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x-5)} = \lim_{x\to 1} \frac{1}{x-5} = \frac{-1}{4}$
3. Graph it, inspect left & righthand limits.

Limits of rational functions at ∞: "big guy over big guy"

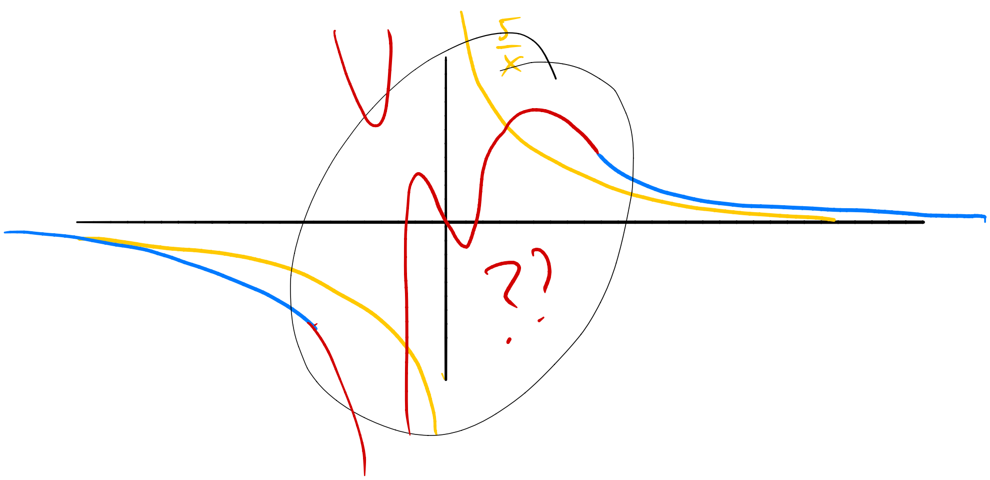
Ex:

- $$\lim_{x\to\infty} \frac{4x^5+3x^3+6}{2x^5-8x} = \lim_{x\to\infty} \frac{4x^5}{2x^5} = \lim_{x\to\infty} \frac{4}{2} = 2$$

- $$\lim_{x\to\infty} \frac{x+2}{x+4} = \lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} 1 = 1$$

- $$\lim_{x\to\infty} \frac{5x^3-2x}{x^4+3x^3+2} = \lim_{x\to\infty} \frac{5x^3}{x^4} = \lim_{x\to\infty} \frac{5}{x} = \text{"}\frac{5}{\infty}\text{"} = 0$$

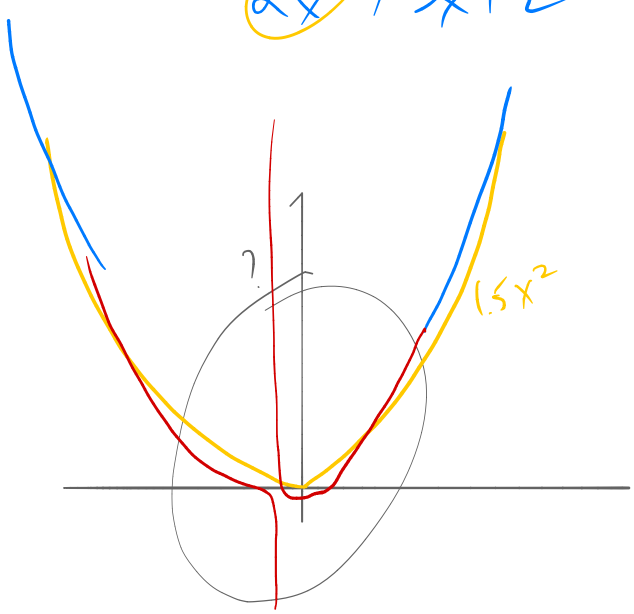
  $$\lim_{x\to-\infty} \frac{5x^3-2x}{x^4+3x^3+2} = \lim_{x\to-\infty} \frac{5x^3}{x^4} = \lim_{x\to-\infty} \frac{5}{x} = \text{"}\frac{5}{-\infty}\text{"} = -0 = 0$$



"$1.5\,\infty^2$"

(DNE)

- $$\lim_{x\to\infty} \frac{3x^5+4x^2-1}{2x^3+3x+2} = \lim_{x\to\infty} \frac{3x^5}{2x^3} = \lim_{x\to\infty} 1.5x^2 = \infty$$



(DNE)

$$\lim_{x\to-\infty} \frac{3x^5}{2x^3} = \lim_{x\to-\infty} 1.5x^2 = \infty$$

"$1.5(-\infty)^2$"

Thurs 9/3

- $\lim_{x\to\infty} \dfrac{\sqrt{9x^6-3x^2+2}}{x^3-2} = \lim_{x\to\infty} \dfrac{\sqrt{9x^6}}{x^3} = \lim_{x\to\infty} \dfrac{3x^3}{x^3} = 3$

(tech. not rational)

$\sqrt{9x^6} = \sqrt{9}\sqrt{x^6} = 3\left(x^6\right)^{1/2} = 3x^3$

☆ Think about $\lim_{x\to\infty}$ as the result of "plugging in $x=\infty$"

Ex:
- $\lim_{x\to\infty} \dfrac{2}{3e^x+4x}$ "=" $\dfrac{2}{\infty} = 0$

- $\lim_{x\to\infty} \dfrac{2x^{50}}{3e^x+4x}$ "=" $\dfrac{\infty}{\infty}$

[fyi: $\dfrac{2x^{50}}{3e^x+4x} \xrightarrow{(\text{as } x\to\infty)} 0$] ← blows up faster than x^{50}

later: L'Hospital rule
need Calculus

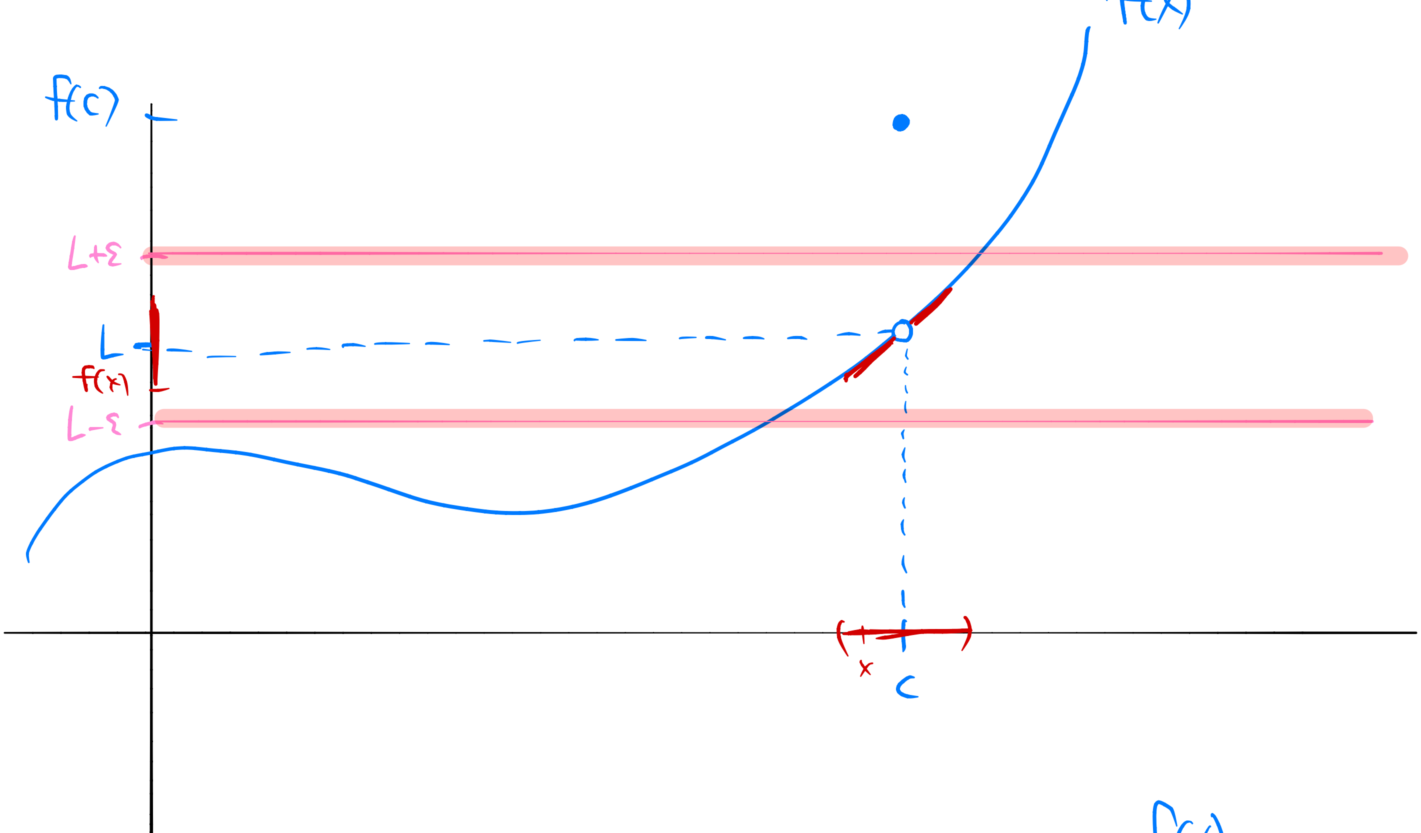
- $\lim_{x\to\infty} \dfrac{\sin x}{x}$ "=" $\dfrac{\text{s.th. b/w } -1 \;\&\; 1}{\infty}$

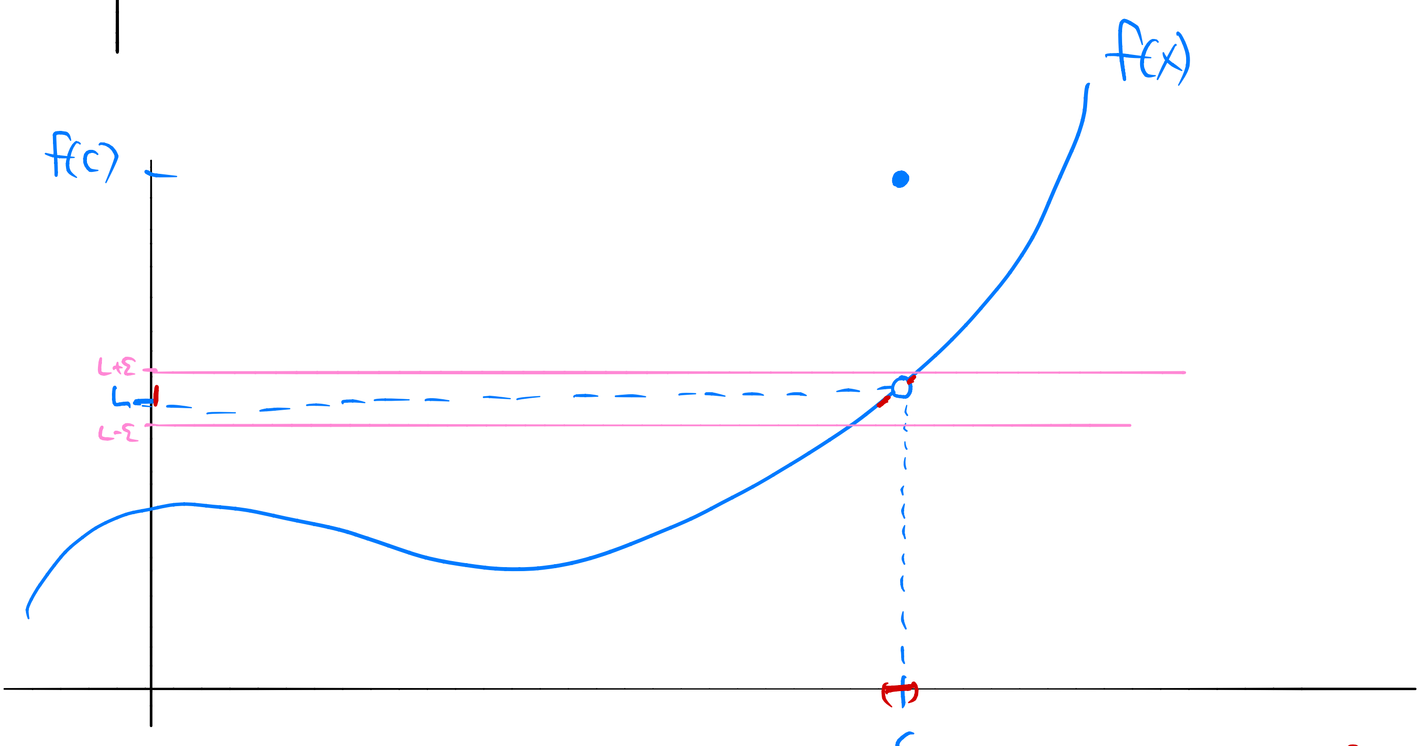
Formal definition of a limit: $\lim_{x \to c} f(x) = L$ means:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $\underbrace{0 < |x-c| < \delta}_{\text{if } x \text{ is w/in dist. } \delta \text{ of } c}$, then $\underbrace{|f(x)-L| < \varepsilon}_{f(x) \text{ is w/in dist } \varepsilon \text{ of } L}$

$\forall$ ↑ "for all"

$\exists$ ↑ there exists





* $\lim_{x \to c} f(x) = L \neq f(c)$, so the limit exists but f(x) is not continuous.