Part 3: Understanding derivatives

Math 1060

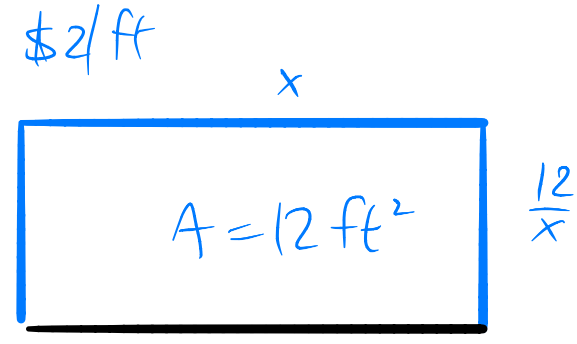
Fall 2020

9/3 – 9/14

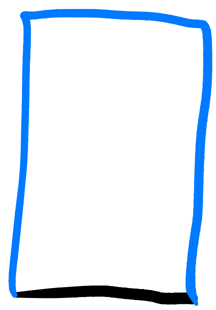
Thurs 9/3

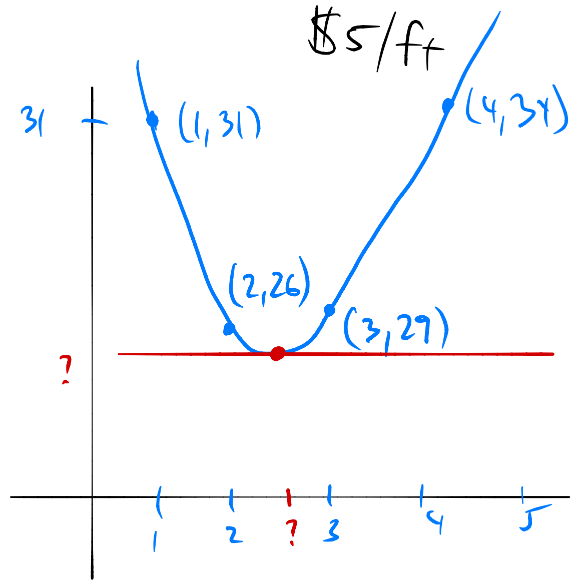
Motivating example, revisited

The cost of building a fence around a $12\,ft^2$ region is $c(x) = \$5x + \$2\left(\frac{12}{x} + x + \frac{12}{x}\right)$
$$= 7x + \frac{24}{x}$$



<u>Note</u>: $c(1) = 7 + 24 = 31$
$c(2) = 14 + 12 = 26$
$c(3) = 21 + 8 = 29$
$c(4) = 28 + 6 = 34$





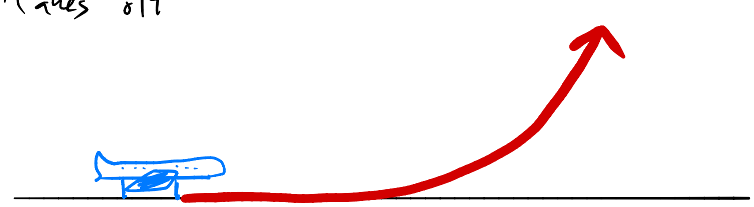
<u>Goal</u>: Minimize $c(x)$

<u>Key observation</u>: The minimum occurs where the <u>tangent line</u> is horizontal, i.e., has slope $m = 0$.

Fri 9/4

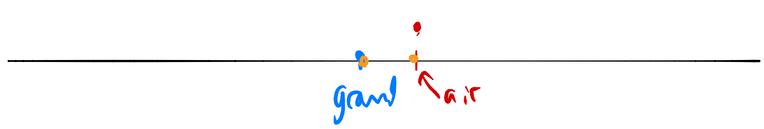
<u>Question</u>: What is the "tangent line"? How to define it? what does its slope represent?

<u>Fun aside</u>: Consider an airplane that takes off



Is there:

- A <u>first</u> point in time that the airplane is in the air?
- A <u>last</u> point in time that the plane is on the ground?



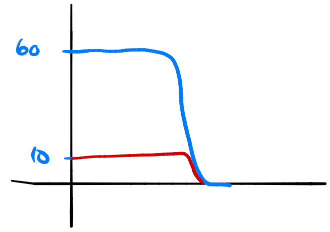
Question: Driving a car...

Can you: go 60 to 0 instantly?

10 to 0 instantly?

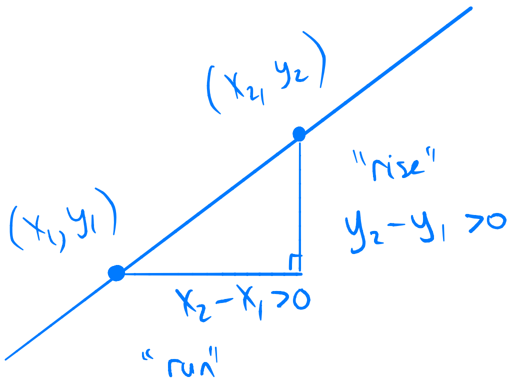
motion to 0 instantly?

If not, how do you stop?



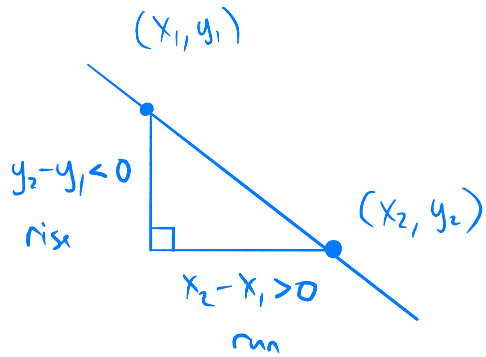
Slope of the line through (x_1, y_1) & (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ if $x_1 \neq x_2$ "rise over run"



slope is positive

line is rising



slope is negative

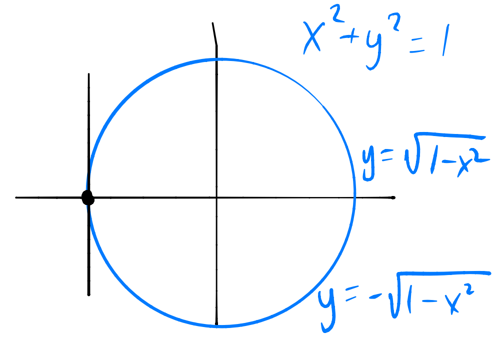
line is falling

slope $= 0 \Rightarrow$ line is flat

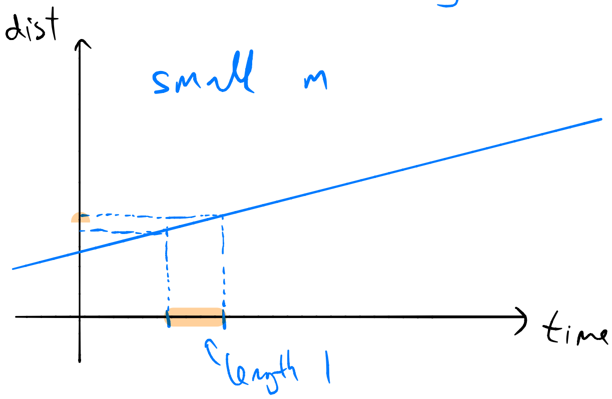
slope = undefined $\Rightarrow$ line vertical

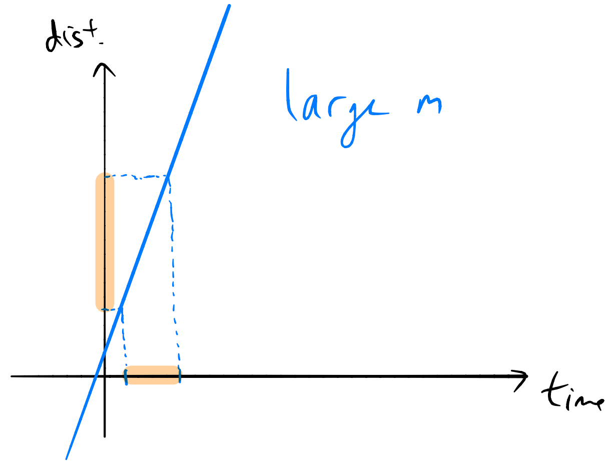
(i.e. $\frac{\text{rise}}{0}$)

impossible for a function, but can occur for curves



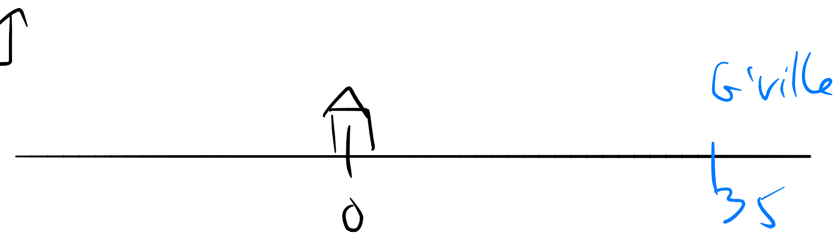
Think about slope as "stretching factor"





Average velocity let $x(t)$ = position of a car at time t.

pos or neg.



The average velocity (rate of change) of $x(t)$ b/w $t=a$ and $t=b$ is $\dfrac{x(b)-x(a)}{b-a} = \dfrac{\text{final pos} - \text{int pos}}{\text{total time}}$

Think about how ave. velocity compares to instantaneous vel.

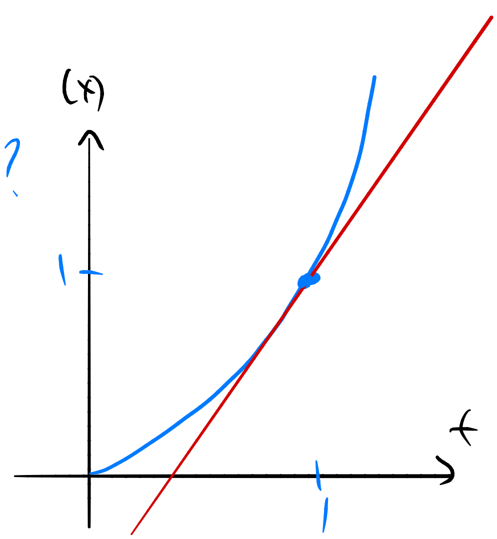
Mon 9/7

Ex: Suppose $x(t) = t^2$ for $0 \le t \le 10$

What is the instantaneous vel. at $t=1$?

At 10 seconds, you have traveled

$x(10) = 10^2 = 100$ meters.

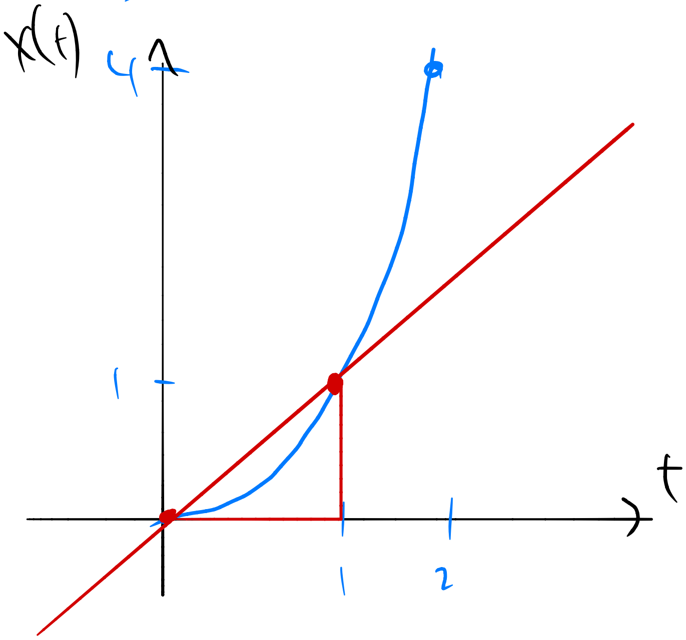


Def: Tangent line intersects @ 1 pt (locally)

Secant line intersects @ 2 pts "

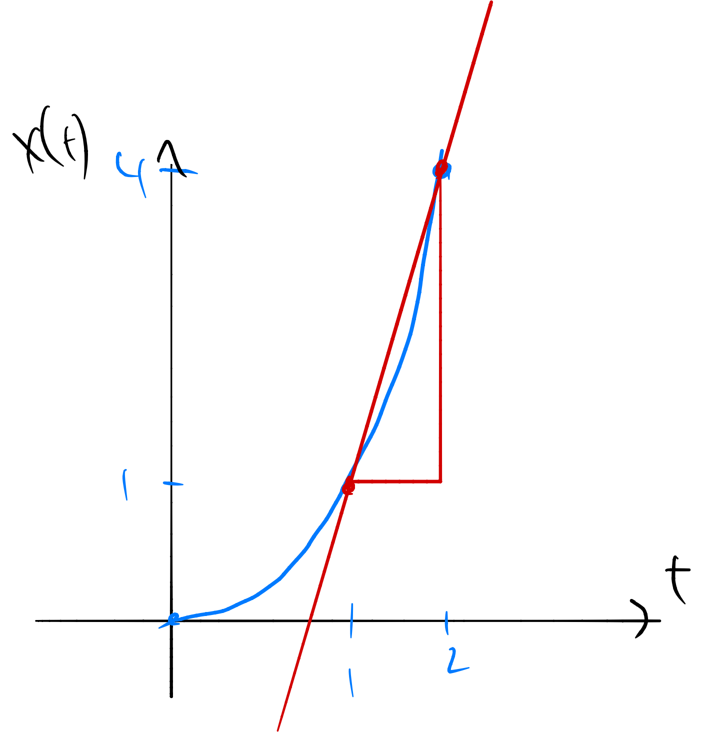
right answer: slope of this line.

Wrong answers



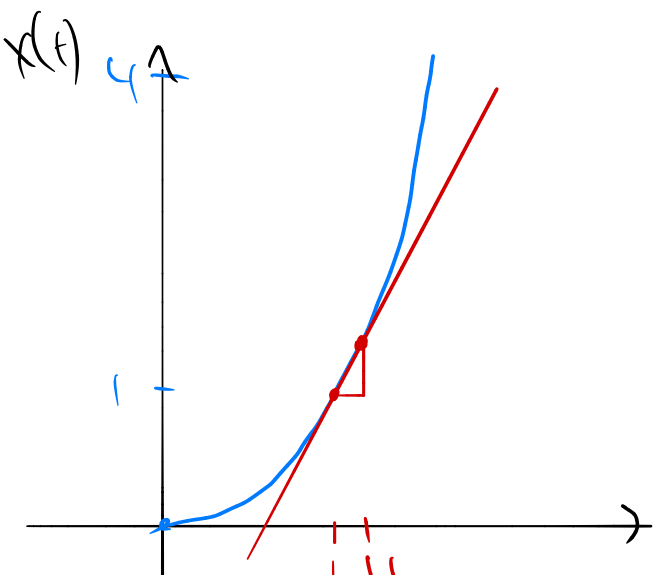
$$\frac{x(1)-x(0)}{1-0} = \frac{1^2-0^2}{1} = 1$$

ave. vel. b/w $t=0$ & $t=1$



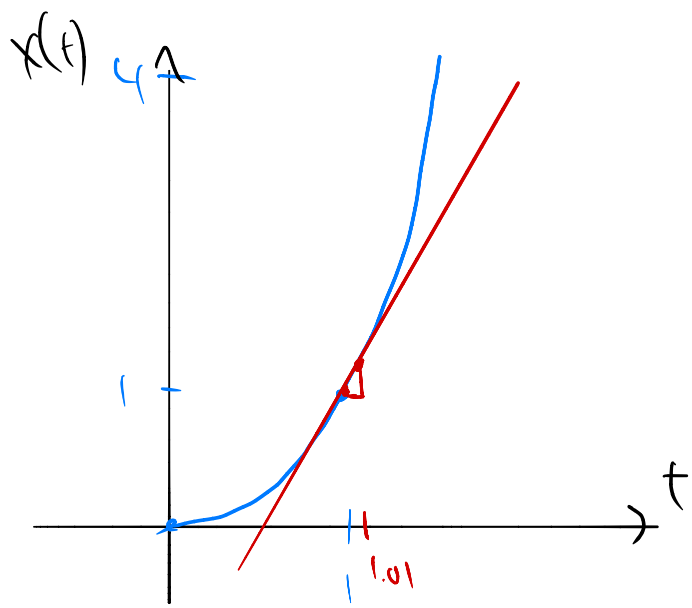
$$\frac{x(2)-x(1)}{2-1} = \frac{2^2-1^2}{1} = 3$$

ave. vel. b/w $t=1$ & $t=2$



$$\frac{x(1.1)-x(1)}{1.1-1} = \frac{1.1^2-1^2}{0.1} = \frac{1.21-1}{0.1} = 2.1$$

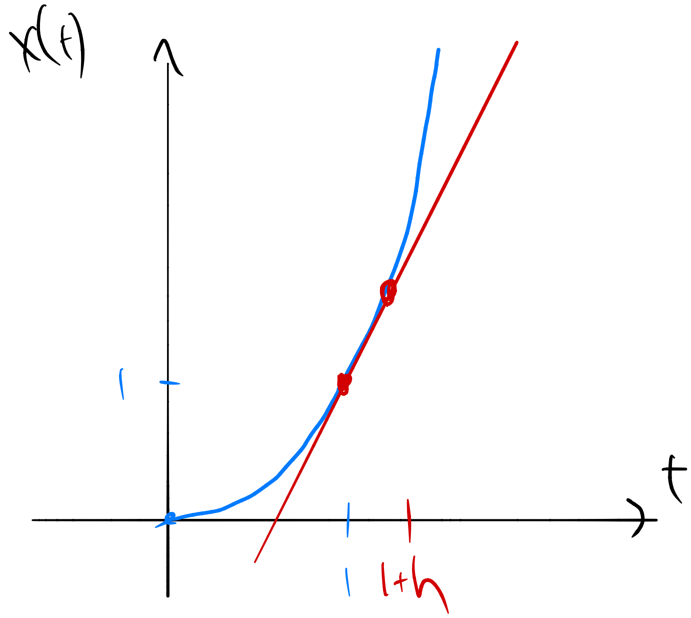
ave. vel. b/w $t=1$ & $t=1.1$



$$\frac{x(1.01)-x(1)}{1.01-1} = \frac{1.01^2-1^2}{0.01} = \frac{1.0201-1}{.01} = 2.01$$

ave. vel. b/w $t=1$ & $t=1.01$

In general: $x(t)=t^2$



ave. velocity b/w $t=1$ & $t=1+h$

$$= \frac{x(1+h)-x(1)}{(1+h)-1} = \frac{(1+h)^2-1}{h} = \frac{\cancel{1}+2h+h^2-\cancel{1}}{h}$$

$$= \boxed{2+h}$$

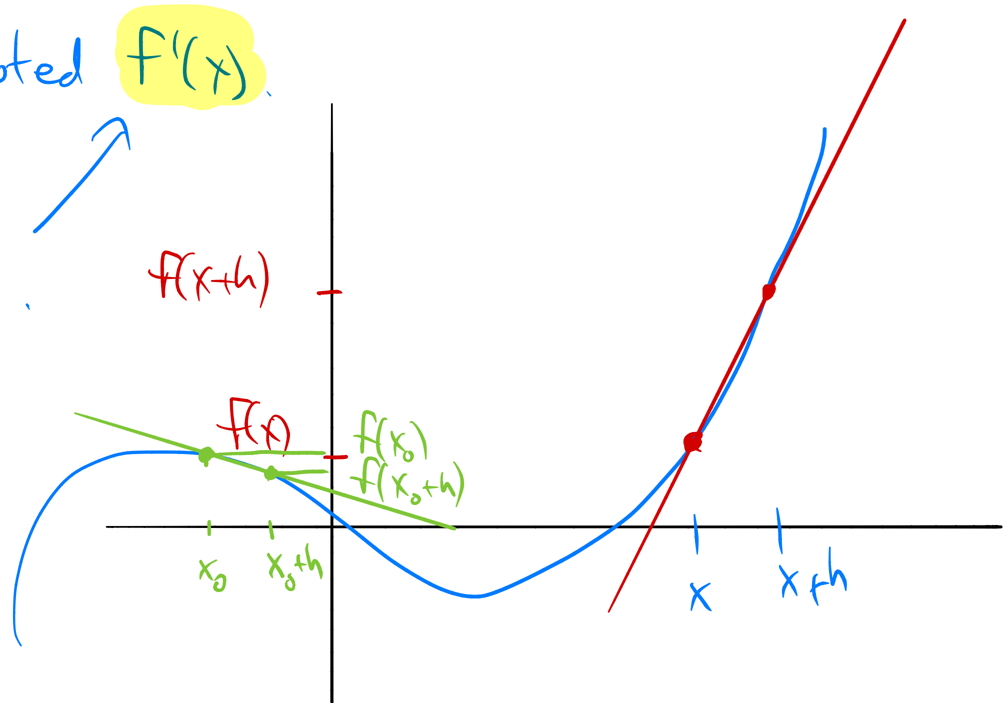
* The instantaneous rate of change at $t=1$ is

$$\lim_{h\to 0}(2+h) = 2.$$

Def: $\overbrace{\text{The instantaneous rate of change}}^{\text{"slope of the tangent line"}}$ of $f(x)$ at x is the function $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, called the derivative of $f(x)$, denoted $f'(x)$.

"f prime".

Wed 9/9



Ex: Compute the derivative of $f(t) = t^2$

input: time
output: position m

Answer: $\lim\limits_{h \to 0} \dfrac{f(t+h) - f(t)}{h} = \lim\limits_{h \to 0} \dfrac{(t+h)^2 - t^2}{h} = \lim\limits_{h \to 0} \dfrac{\cancel{t^2} + 2th + h^2 - \cancel{t^2}}{h}$

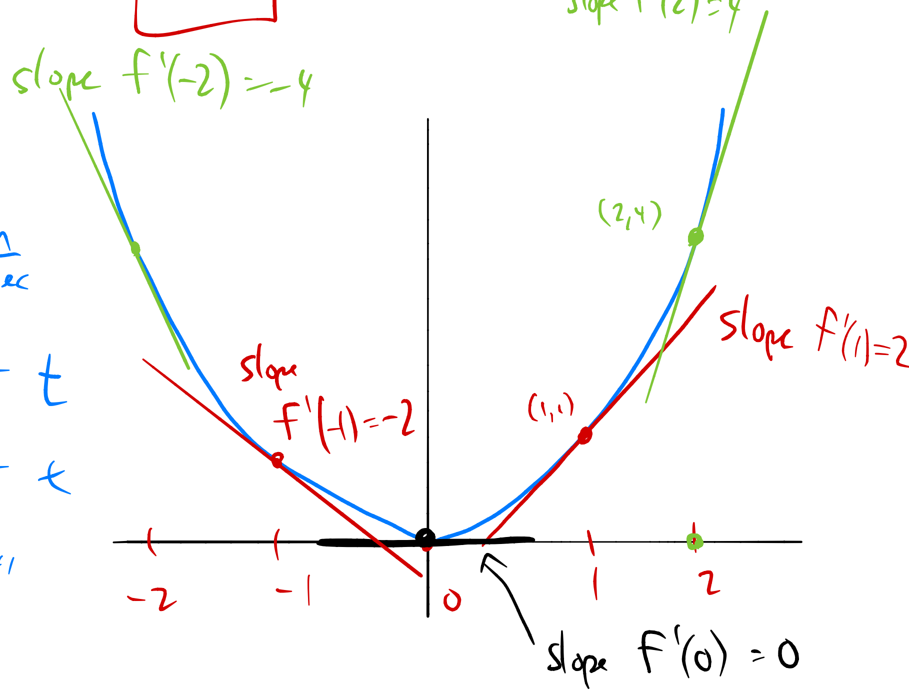
$= \lim\limits_{h \to 0} \dfrac{2th + h^2}{h} = \lim\limits_{h \to 0} 2t + h = \boxed{2t}$

Graphically:

$f'(t) = 2t$

input: time
output velocity $\frac{m}{sec}$

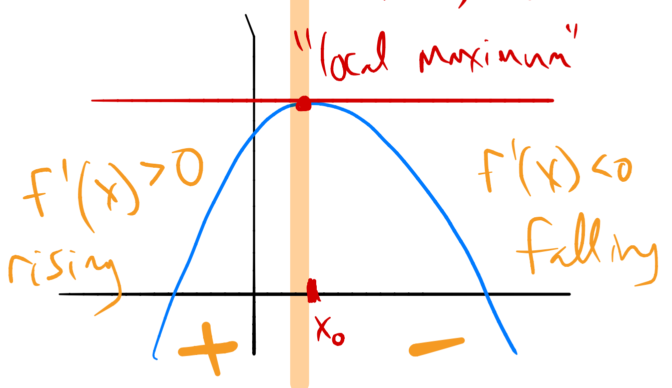
slope of tangent line at t
= instan. rate of change at t
= "vel. of golf cart at t"

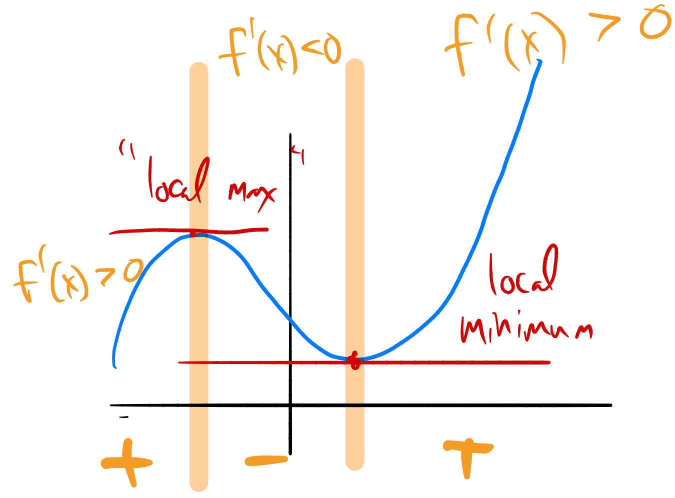


The interplay b/w a function & its derivatives

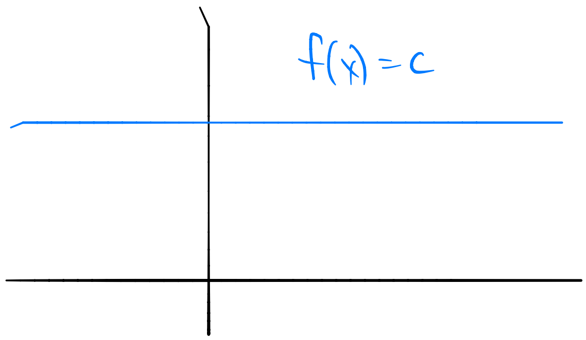
- $f'(x) > 0$ means $f(x)$ is increasing or "rising"
- $f'(x) < 0$ means $f(x)$ is decreasing or "falling"
- $f'(0) = 0$ is a critical point

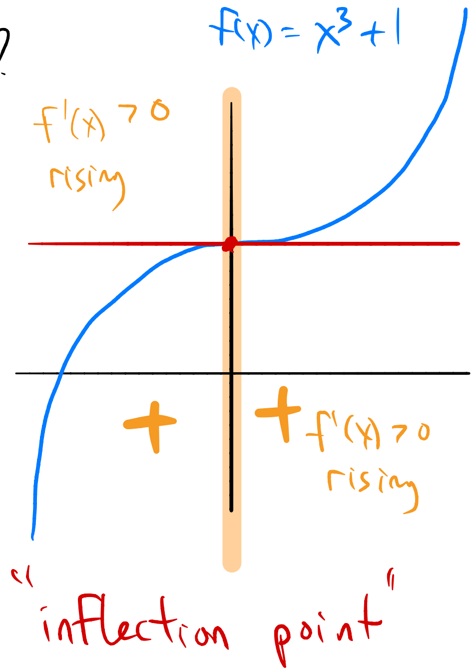
Critical points: $f'(x_0) = 0$

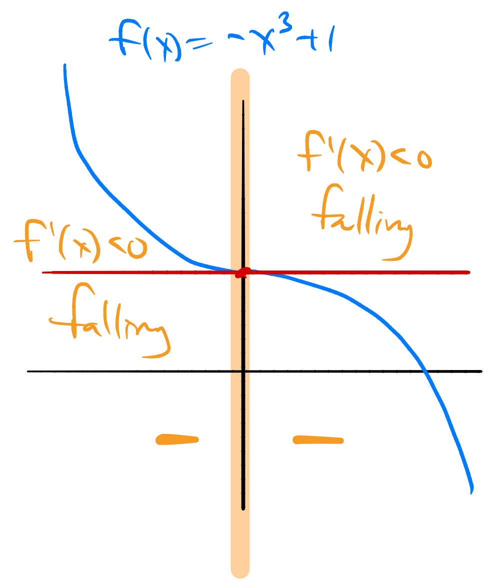




other types of critical pts?

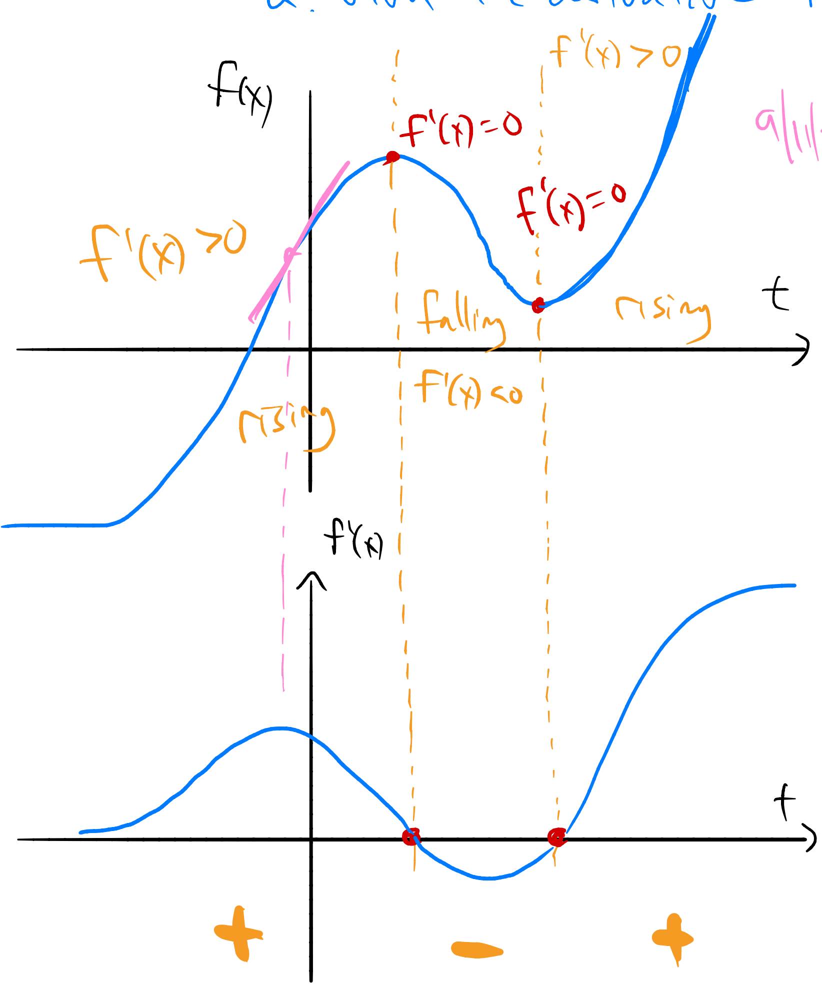


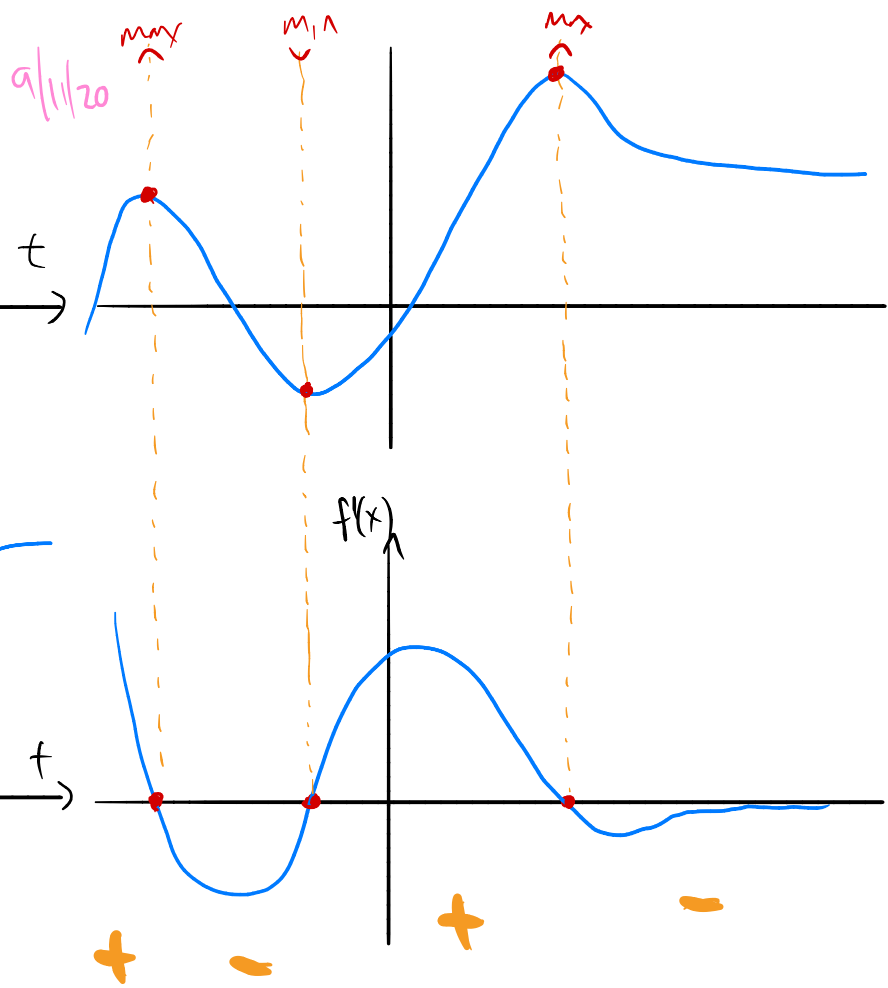




Exercise: 1. Given a function $f(x)$, graph its derivative, $f'(x)$

2. Given the derivative $f'(x)$, graph the function.





Second derivative

Motivating example: Let $x(t)$ = pos. of car (or ball) at time t

Then $x'(t)$ = rate of change of position

= velocity $v(t)$.

Note: $v'(t)$ is "rate of change of velocity" = acceleration.

This is the second derivative of $x(t)$, write $f''(t)$

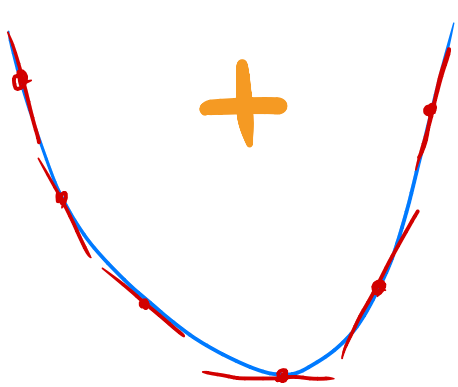
"rate of change of the rate of change of $x(t)$"

Say: "f double prime of t"

What does the second derivative measure?

$f'' > 0$

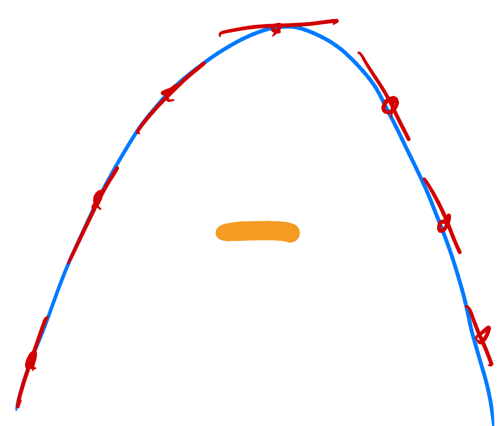
slope of tangent line is increasing



"Concave up"

$f'' < 0$

slope of tangent line is decreasing



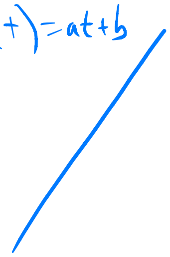
"Concave down"

What if $f''=0$?

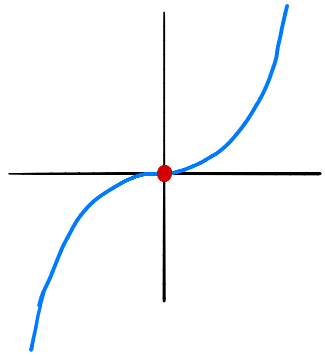
Ex:

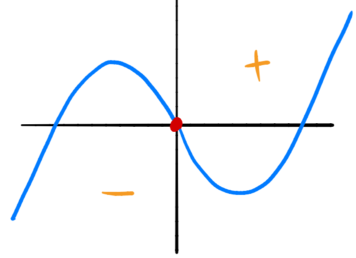
$f(t)=c$

$f(t)=at+b$

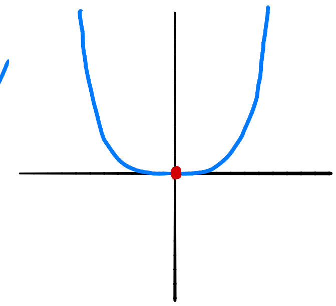


$f(t)=t^3$





$f(t)=t^4$



"threshold b/w concave up & concave down"

problem case

Moral: If $f''=0$, we don't necessarily know about concavity.

(the "inconclusive case")

Tangent line approximation

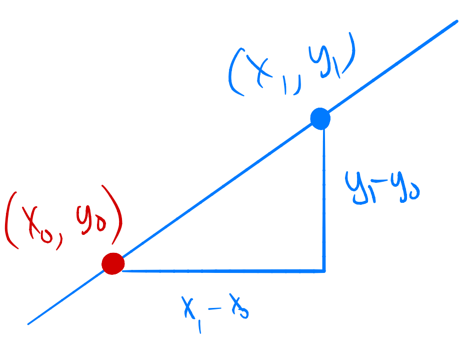
Goal: (1) Find the equation of the tangent line, not just its slope, at $x=a$

i.e., $y = mx + b$

slope ↑ ↑ ???

(2) Use this to approximate $f(x)$ for $x \approx a$

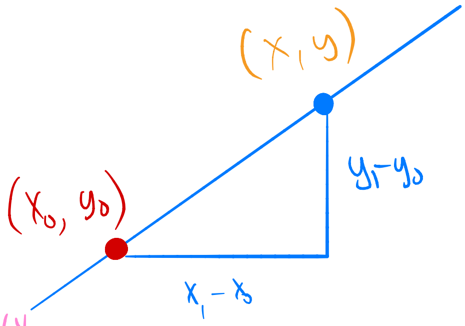
Recall: Formula for a line through (x_0, y_0) with slope m



$y = mx + b$ ... how to find m & b?

slope $m = \dfrac{y_1 - y_0}{x_1 - x_0}$

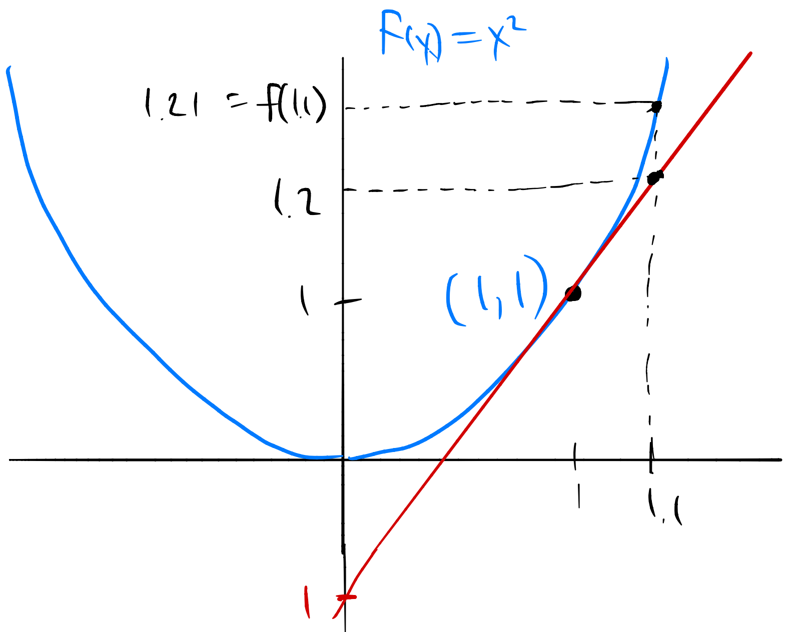
$y = mx + b$ ... how to find m & b?



slope $m = \dfrac{y - y_0}{x - x_0} \Rightarrow \boxed{y - y_0 = m(x - x_0)}$

Mon 9/14

Exercise: Use the tangent line to approximate $f(x) = x^2$ at $x = 1$.



$x_0 = 1, \quad y_0 = f(x_0) = f(1) = 1$

$f'(x) = 2x, \quad f'(x_0) = f'(1) = 2$

$y - y_0 = m(x - x_0)$

$y - 1 = 2(x - 1)$

$y = 2x - 1$ tangent line

Approximation: plug in $x_0 = 1.1$

$y = 2(1.1) - 1 = 2.2 - 1 = 1.2$

i.e., $f(1.1) = 1.1^2 \approx 1.2$

Approximating functions with polynomials is how computers evaluate them.

For example, $f(x) = \sqrt{x}$. How would you compute $f(x) = \sqrt{20}$?

We just did a linear (1st order) approximation.

Computers can easily do 10th order, or 100th order, etc.

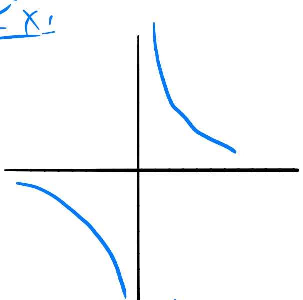
others: $f(x) = e^x$, 1.5^x, $\ln x$, $\sin \frac{1}{2}$

This is a topic in Calc 2.

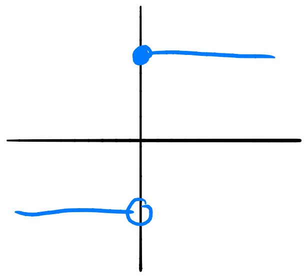
Sometimes, there is no linear approx at x_0. This occurs if you zoom in & the function doesn't look like a straight line.

Ex:

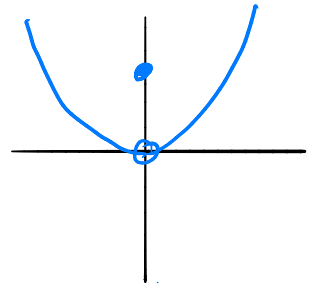
$f(x) = \sin\frac{1}{x}$



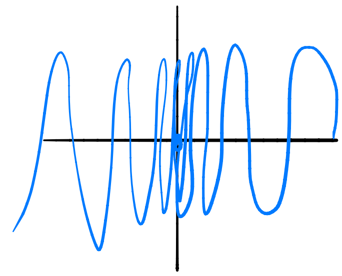
not defined at $x=0$



not continuous at $x=0$

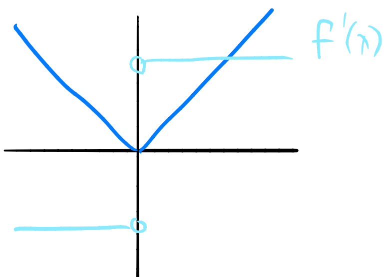


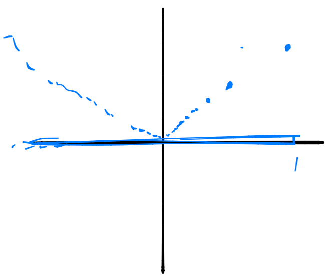
not contin. at $x=0$



not locally like a line at $x=0$

$f(x) = |x|$





$$f(x) = \begin{cases} 0 & x=0 \\ 0 & x \text{ irrat.} \\ 1/b & x = \frac{a}{b} \quad a \neq 0 \end{cases}$$

continuous at $x=0$

not "differentiable" at $x=0$

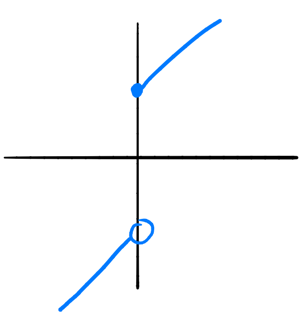
How to characterize this?

$f(x)$ is continuous at $x=a$ if $\lim_{x\to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

differentiable at $x=a$ if ① $f(x)$ is continuous.

② $\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x)$

Example of how ② could hold but ① could fail



$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 1$$