Part 4: Computing derivatives

Math 1060
Fall 2020
9/16 - 9/29

Wed 9/16

Notation: $(f(x))'$ for $f'(x)$. e.g., $\left(\dfrac{x^2+3x-2}{\sin 2x}\right)'$

Goal: We want formulas for computing derivatives, b/c the limit defn is tedious.

Polynomials

- $f(x)=x^n$, $n=0,1,2,\ldots$

$\lim_{h\to 0}\frac{h^2}{h}=0$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to 0}\frac{[\cancel{x^n}+nx^{n-1}h+h^2(\_\_\_\_)]-\cancel{x^n}}{h}$$

$$=\lim_{h\to 0}\frac{nx^{n-1}\cancel{h}}{\cancel{h}}=\lim_{h\to 0}nx^{n-1}=\boxed{nx^{n-1}}$$

$$\underbrace{(x+h)(x+h)(x+h)\cdots(x+h)}_{n\text{ times}}=x^n+nx^{n-1}h^1+\_\_x^{n-2}h^2+\cdots+nx^1h^{n-1}+h^n$$

$(x+h)(x+h)=x^2+hx+xh+h^2$

$(x+h)(x+h)(x+h)=(x^2+hx+xh+h^2)(x+h)$
$=x^3+3x^2h+3xh^2+h^3$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
⋮
1 n n 1

Examples: $(x^{10})'=10x^9$, $(x^3)'=3x^2$, $(x^2)'=2x^1$, $(x^1)'=1x^0=1$

$(1)'=(x^0)'=0x^{-1}=0$

- Derivatives of sums Given f, g, what is $(f+g)'$?

e.g. $(x^{10}+x^3)'\stackrel{?}{=}(x^{10})'+(x^3)'$

$=10x^9+3x^2$ Yes!

$$(f+g)' = \lim_{h\to 0} \frac{\boxed{f(x+h)+g(x+h)} - \boxed{f(x)+g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x).$$

- Derivatives & scalar multiplication. Given f, what is $(cf)'$

e.g., $(5x^{10})' = 5(x^{10})' = 5\cdot 10x^9 = 50x^9$?

$$(cf)' = \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= c f'(x)$$

$$\frac{b}{b}\frac{1}{a} - \frac{1}{b}\frac{a}{a} = \frac{\quad}{ab}$$

Now, we can differentiate polynomials

e.g. $(x^5+4x^3+18x^2-10)' = 5x^4+12x^2+36x^1$

Careful! $(f(x)\,g(x))' \neq f'(x)\cdot g'(x)$

Fri 9/18

Goal for today: learn how to compute $(f(x)\,g(x))'$, $\left(\frac{f(x)}{g(x)}\right)'$

- Reciprocal rule: $\left(\frac{1}{f}\right)' = \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$

$$= \lim_{h\to 0}\left[\frac{1\cdot f(x)}{f(x+h)\,f(x)} - \frac{1\cdot f(x+h)}{f(x)\,f(x+h)}\right]\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{f(x) - f(x+h)}{f(x+h)\,f(x)} \cdot \frac{1}{h}$$

$$= \underbrace{\boxed{\lim_{h\to 0} \frac{[f(x+h) - f(x)]}{h}}}_{f'(x)} \cdot \frac{-1}{f(x+h)\,f(x)}$$

$$= f'(x) \cdot \frac{-1}{(f(x))^2} = -\frac{f'(x)}{(f(x))^2}$$

Example

1. $$\left(\frac{1}{x^5+4x^3+18x^2-10}\right)' = \frac{-(5x^4+12x^2+36x)}{(x^5+4x^3+18x^2-10)^2}$$

2. Recall $(x^n)' = n x^{n-1}$ for $n = 0, 1, 2, 3, \ldots$

e.g., $(x^4)' = 4x^3$

what is $(x^{-4})' = \left(\frac{1}{\underbrace{x^4}_{f(x)}}\right)' = -\frac{f'(x)}{(f(x))^2} = \frac{-4x^3}{x^8} = \frac{-4}{x^5} = \boxed{-4x^{-5}}$

More generally, say $f(x) = x^n$, $f'(x) = n x^{n-1}$

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = -\frac{f'(x)}{(f(x))^2} = \frac{-n x^{n-1}}{x^{2n}} = \boxed{-n x^{-n-1}} = \frac{-n}{x^{n+1}}$$

Remark: $(x^n)' = n x^{n-1}$ holds for all integers (pos or neg)

- Product rule   Wrong: $\cancel{(f\cdot g)' = f' g'}$

$$(f\cdot g)' = \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \boxed{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}} \cdot g(x+h) + \boxed{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}} \cdot f(x)$$

$$= f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

"Product rule": $(f(x)\,g(x))' = f'(x)\,g(x) + f(x)\,g'(x)$

- Quotient rule: $\left(\frac{f(x)}{g(x)}\right)'$ Note: $\frac{f}{g} = f \cdot \frac{1}{g}$

$$\Rightarrow \left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \frac{-g'}{g^2}$$

$$= \frac{f'g}{g^2} - \frac{fg'}{g^2} = \boxed{\frac{f'g - fg'}{g^2}}$$

This is more general than the recip. rule
← special case of $f = 1$

Mon 9/21

Notation for derivatives

Lagrange (Italy, 1736–1813): $f'(x)$

Euler (Switzerland, 1707–83): Df

Newton (England, 1643–1727): $\dot{y}$ adopted in Britian

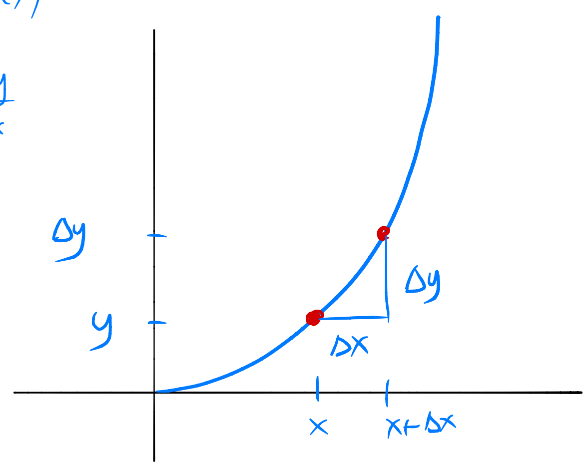
Leibniz (Germany, 1646–1716): $\frac{dy}{dx}$ adopted in Europe

Advantages of Leibniz's notation resulted in Britian falling behind 100–200 yrs to mainland Europe, mathematically.

Leibniz's notation: If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$

Motivating example: Slope of secant line is $\frac{\Delta y}{\Delta x}$

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$

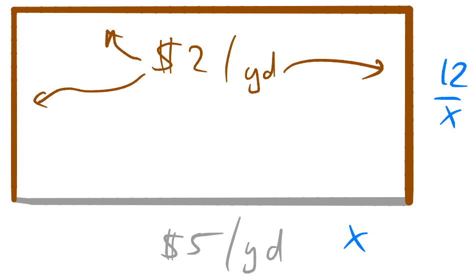


Notational reasoning: Ancient Greek symbol Δ had evolved "in the limit" to the modern "d"

Another way to interpret this: $\frac{dy}{dx} = \frac{d}{dx}(y)$

Second derivative: $\frac{d}{dx}\left(\frac{d}{dx}(y)\right) = \frac{d^2y}{dx^2}$

Application to problem from Day 1:



Cost of Fence is $c(x) = 7x + \frac{48}{x} = 7x + 48x^{-1}$

Goal: Find min of $c(x)$.

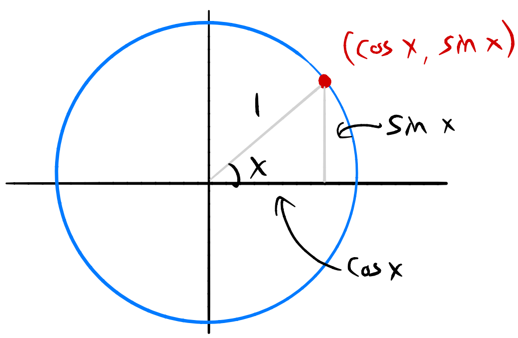
$$c'(x) = 7 - 48x^{-2} = 0$$

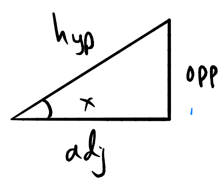
$$7 - \frac{48}{x^2} = 0 \Rightarrow x^2 = \frac{48}{7} \Rightarrow x = \pm\sqrt{\frac{48}{7}} \approx 2.619$$

$$\Rightarrow \text{min cost is } c\left(\sqrt{48/7}\right) \approx 36.661$$

Next goal: Compute derivatives of trig functions.

Quick review:





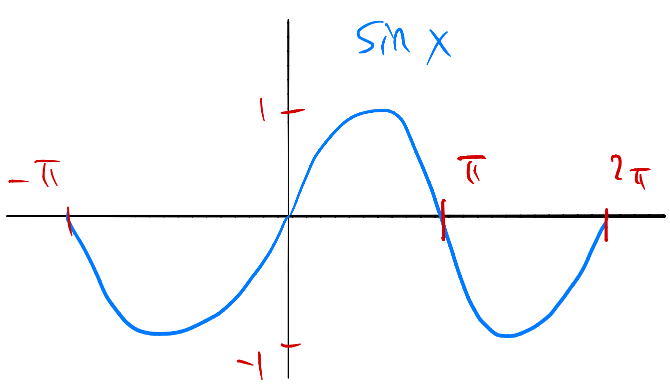
$\sin x = \frac{\text{opp}}{\text{hyp}}$

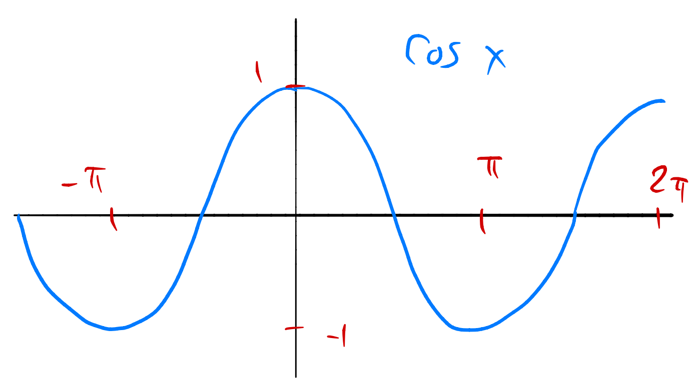
$\cos x = \frac{\text{adj}}{\text{hyp}}$

$\tan x = \frac{\sin x}{\cos x} = \frac{\text{opp}}{\text{adj}}$.

Also, $\frac{1}{\sin x} = \csc x$, $\frac{1}{\cos x} = \sec x$, $\frac{1}{\tan x} = \cot x$.

Wed 9/23



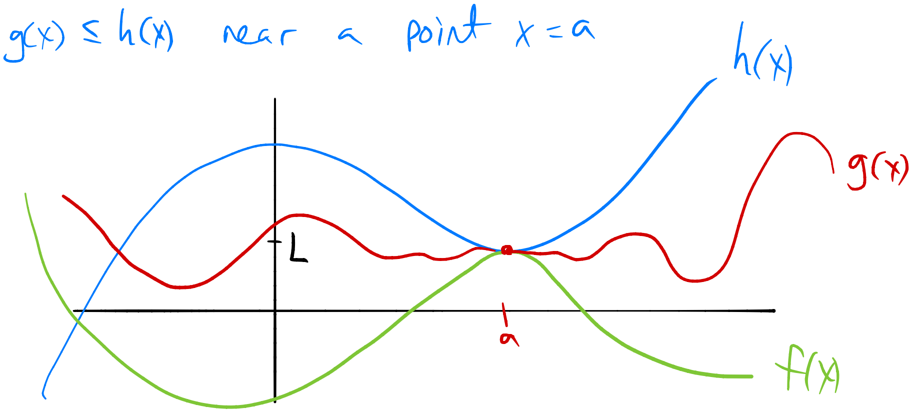


We'll need the following result to compute $\lim_{x\to 0} \frac{\sin x}{x}$ and $\lim_{x \to 0} \frac{\cos x - 1}{x}$, which in turn we'll need to differentiate $\sin x$ & $\cos x$.

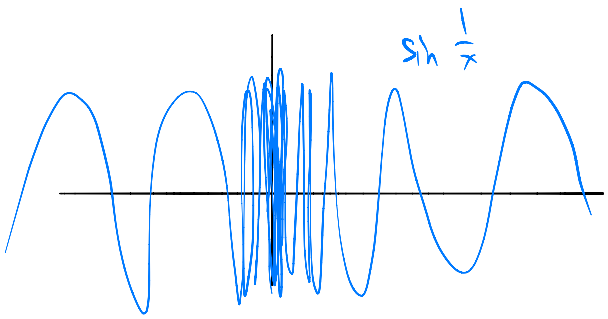
Squeeze theorem: Suppose $f(x) \le g(x) \le h(x)$ near a point $x = a$

If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

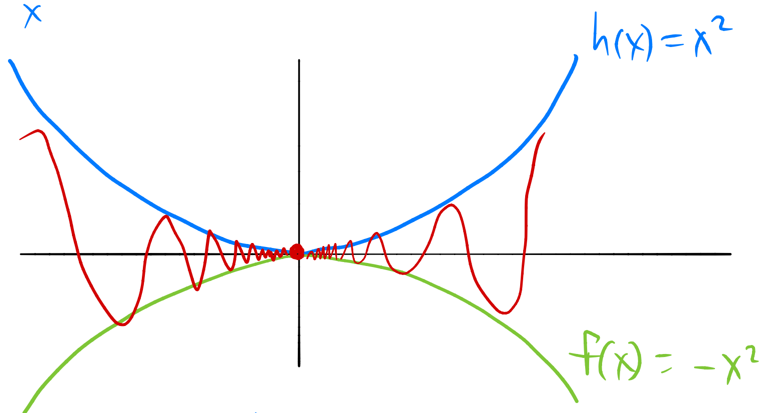
then $\lim_{x \to a} g(x) = L$



Example application: Compute $\lim_{x \to 0} x^2 \sin \frac{1}{x}$



$\lim_{x \to 0} \sin \frac{1}{x}$ DNE



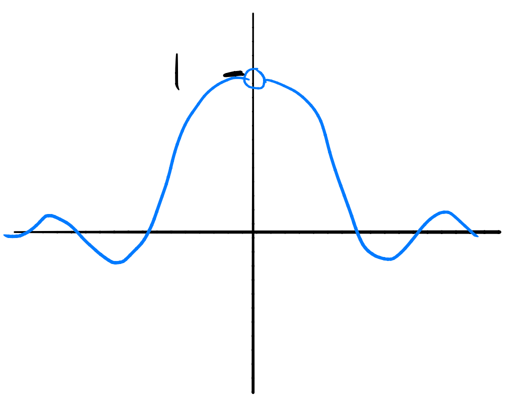
$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$ by squeeze theorem.

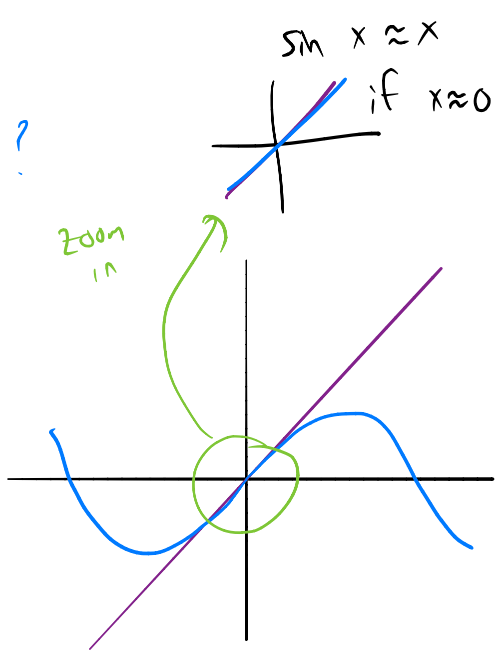
Formally:

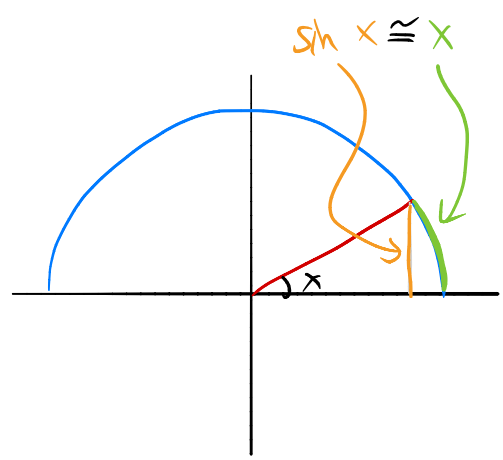
$$\begin{array}{ccccc} -x^2 & \le & x^2 \sin \frac{1}{x} & \le & x^2 \\ \text{as } x \to 0 \downarrow & & \downarrow \text{sq. thm} & & \downarrow \text{as } x \to 0 \\ 0 & & 0 & & 0 \end{array}$$

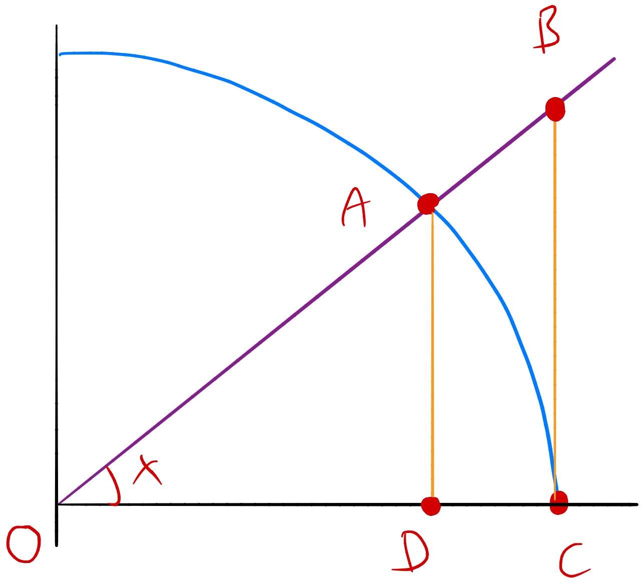
• Compute $\lim_{x \to 0} \frac{\sin x}{x}$

First, what "should" it be?









area ΔAOD < area AOD < area ΔBOC

$$\frac{1}{2}\sin x\cos x \le \frac{x}{2} \le \frac{1}{2}\tan x$$

$$\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

$$\frac{1}{\cos x} \ge \frac{\sin x}{x} \ge \cos x$$

$$\frac{1}{\cos x} \xrightarrow{\lim_{x\to 0}} 1 \qquad \cos x \xrightarrow{\lim_{x\to 0}} 1$$

By squeeze theorem, $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Exercise: Show that $\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$

$$\boxed{\text{Derivatives of trig functions}}$$

- $$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{[\cos x\sin h + \sin x\cos h] - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\sin h}{h} + \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h}$$

$$= \lim_{h\to 0}\cos x \lim_{h\to 0}\frac{\sin h}{h} + \lim_{h\to 0}\sin x \lim_{h\to 0}\frac{\cos h - 1}{x}$$

$$= \cos x \cdot 1 + \sin x \cdot 0 = \boxed{\cos x}$$

$$\bullet\ (\cos x)' = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{[\cos x \cos h - \sin x \sin h] - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$

$$= \lim_{h\to 0} \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1 = \boxed{-\sin x}$$

Fri 9/25

$$\bullet\ (\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2}$$

$$= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2} = \boxed{(\sec x)^2}$$

$$\bullet\ (\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{0 \cos x + 1 \cdot \sin x}{(\cos x)^2} = \boxed{\tan x \sec x}$$

Summary

toggle "co-",
mult. by -1

$(\sin x)' = \cos x \longrightarrow (\cos x)' = -\sin x$

$(\tan x)' = (\sec x)^2$ $\qquad$ $(\cot x)' = -(\csc x)^2$

$(\sec x)' = \sec x \tan x$ $\qquad$ $(\csc x)' = -\csc x \cot x$

Chain rule

Motivation with old example:

$W = \frac{12}{L}$

$L = \frac{12}{W}$

Cost:

$$C(L) = 7L + \frac{48}{L}$$

$$C(W) = 7\left(\frac{12}{W}\right) + \frac{48}{12/W}$$

$$= \frac{84}{W} + 4W$$

$$L(W) = \frac{12}{W}$$

How are $\frac{dC}{dL}$, $\frac{dL}{dW}$, $\frac{dC}{dW}$ related?

$$\frac{dC}{dL} = 7 - 48L^{-2}, \quad \frac{dL}{dW} = -12W^{-2}, \quad \frac{dC}{dW} = 4 - 84W^{-2}$$

$$= 7 - \frac{48}{144/W^2}$$

Note: $\frac{dC}{dL} \cdot \frac{dL}{dW} = \frac{dC}{dW}$

Analogy:

- Clemson scores 3x as fast as Bama
- Bama scores 12x as fast as USC

$\Rightarrow$ Clemson scores 36x as fast as USC.

That is, $\dfrac{d\,\text{Clemson}}{d\,\text{USC}} = \dfrac{d\,\text{Clemson}}{d\,\text{Bama}} \cdot \dfrac{d\,\text{Bama}}{d\,\text{USC}}$

$$36 = 3 \cdot 12$$

Chain rule The derivative $\dfrac{d}{dx}\big(f(g(x))\big)$ is

$$[f(g(x))]' = f'(g(x))\,g'(x) \qquad \frac{df}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx}$$

Lagrange $\qquad$ Leibnitz.

Mon 9/28

Examples:

- $(\sin 2x)' = \dfrac{d}{dx}(\sin 2x)$

  Write as $f(x) = \sin x \qquad f'(x) = \cos x$

  $g(x) = 2x \qquad g'(x) = 2$

  $(f(g(x)))' = f'(g(x)) \cdot g'(x) = \cos 2x \cdot 2 = 2\cos 2x$

- $[(\cos x^3)'] = \dfrac{d}{dx}(\cos x^3)$

  Write as $f(x) = \cos x \qquad f'(x) = 3x^2$

  $g(x) = x^3 \qquad g'(x) = -\sin x$

  $(f(g(x)))' = f'(g(x)) \cdot g'(x) = \sin x^3 \cdot 3x^2 = 3x^2 \cdot \sin x^3$

Compare to $[(\cos x)^3]' = 3(\cos x)^2 \cdot (-\sin x)$

- $\left[(6x^3+7x)^8\right]' = \frac{d}{dx}(6x^3+7x)^8$

Let's compare/contrast both notations:

<u>Lagrange</u>

$f(x) = x^8 \qquad f'(x) = 8x^7$

$g(x) = 6x^3+7x \qquad g'(x) = 18x^2+7$

$$f(g(x))' = f'(g(x)) \cdot g'(x)$$
$$= 8(6x^3+7x)^7 \cdot (18x^2+7)$$

<u>Leibnitz:</u>

Let $y = u^8$, where $u = 6x^3+7x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= 8u^7 \cdot (18x^2+7)$$
$$= 8(6x^3+7x) \cdot (18x^2+7)$$

<u>Useful identities</u>

$(\sin kx)' = k \cos kx$

$(\cos kx)' = -k \sin kx$

$\boxed{\text{Implicit differentiation}}$

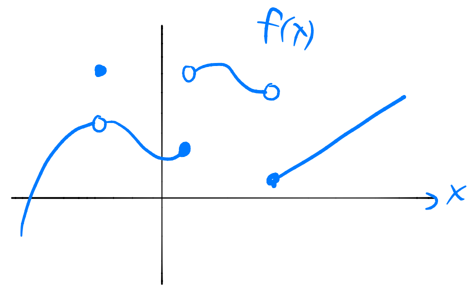
What's the difference b/w defining a function "explicitly" vs. "implicitly"?

- Functions defined <u>explicitly</u>

<u>Ex:</u> $f(x) = \sin 2x$

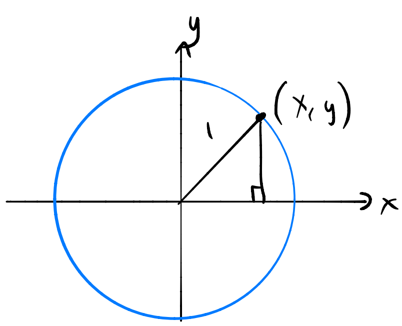
$y = x^2 - \cos 3x + 2$

$$y(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



- Functions defined <u>implicitly</u>:

<u>Ex:</u> $F(x) = x^2 + y^2 = 1$



In this case, we can actually <u>solve</u> for y:

$y = \pm\sqrt{1-x^2}$, so it's actually 2 functions.

But frequently, we can't solve for y.

For example, consider $xy + \sin y = \frac{1}{\sqrt{xy}} - 1$.

Wed 9/30

* key point: Even for implicitly defined functions, we can still find the derivative $\frac{dy}{dx}$

Method: Differentiate both sides & solve for $\frac{dy}{dx}$

Ex 1: $xy + x\sin y = 3x$

$$(xy)' + (x\sin y)' = (3x)'$$

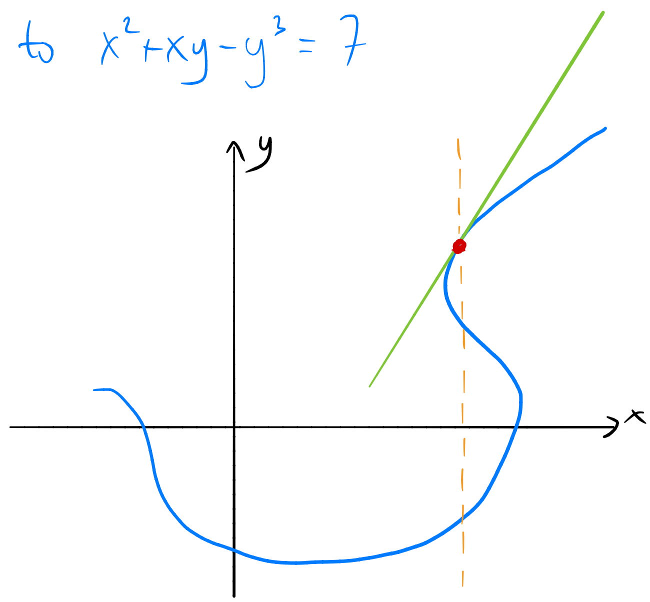
$$\left(1\cdot y + x\frac{dy}{dx}\right) + \left(1\cdot \sin y + x\cos y\cdot\frac{dy}{dx}\right) = 3$$

Note: x is a variable, y is a function

$$(x + x\cos y)\frac{dy}{dx} + y + \sin y = 3$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{3 - y - \sin y}{x + x\cos y}}$$

Ex 2: Find the equation of the line tangent to $x^2 + xy - y^3 = 7$ at the point $(x_0, y_0) = (3, 2)$



Note: This is not a function (fails vertical line test), so we prefer to write $\frac{dy}{dx}$ to $y'(x)$.

$$\frac{d}{dx}(x^2 + xy - y^3) = \frac{d}{dx}(7)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) - \frac{d}{dx}(y^3)$$

$$2x + y + x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$2x + y = (3y^2 - x)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{2x+y}{3y^2 - x}} \Rightarrow \left.\frac{dy}{dx}\right|_{(x,y)=(3,2)} \frac{2x+y}{3y^2-x} = \frac{2\cdot 3 + 2}{3\cdot 2^2 - 3} = \frac{8}{9}$$

Tangent line: $y - y_0 = m(x - x_0) \Rightarrow \boxed{y - 2 = \frac{8}{9}(x - 3)}$

$y_0 = 2$, $m = 8/9$, $x_0 = 3$