Part 6: Understanding integrals

Math 1060

Fall 2020

10/9 - 10/21

Fri 10/9

Previously: we've studied differential calculus – derivatives & rates of change.

Given a function $f(x)$, find its derivative, $f'(x)$.

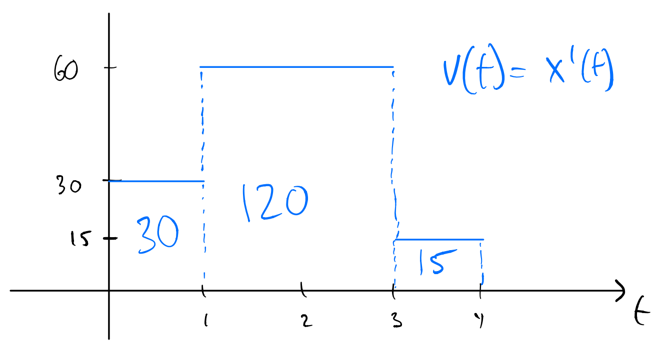
Next, we'll do integral calculus, which is the opposite.

Given a rate $f'(x)$, find the "antiderivative" $f(x)$.

Big idea: Function $f(x) \underset{\text{antiderivative}}{\overset{\text{take derivative}}{\rightleftarrows}} f'(x)$ "rate of change"

area under curve ← We'll motivate this next.

Motivating example: Consider a 4-hour road trip, where the velocity you travel is the following:



Question: How far did you travel?

There are two ways to answer.

Method 1: "area under curve"

$$0 \le t \le 1 \quad \left(30 \tfrac{mi}{hr}\right)(1\ hr) = 30\ mi$$
$$1 \le t \le 3 \quad \left(60 \tfrac{mi}{hr}\right)(2\ hr) = 120\ mi$$
$$3 \le t \le 4 \quad \left(15 \tfrac{mi}{hr}\right)(1\ hr) = 15\ mi$$

total dist. $= 30 + 120 + 15 = 165\ mi$

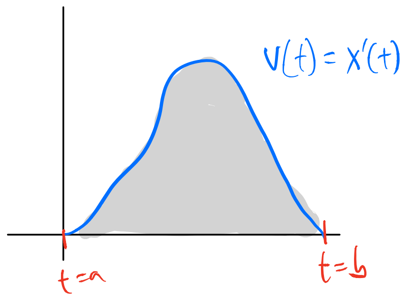
Method 2 "odometer"

Check your odometer after & before the trip. Subtract these values.

$$x(4) - x(1) = \boxed{5\,2\,0\,0} - \boxed{5\,0\,3\,5} = 165\ mi$$

★ This is half of the "Fundamental Theorem of Calculus."

It works more generally, not just for piecewise functions.



<u>Velocity</u> is the derivative of <u>distance</u>

Total distance is:

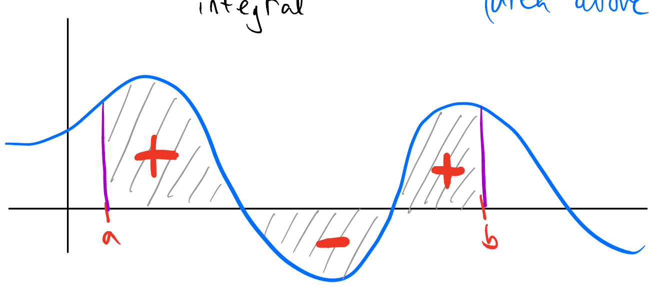
- Area under the curve of $x'(t)$
- $x(b) - x(a)$

<u>Key concept</u>: The "net area", or "signed area" from a to b,

denoted $\int_a^b f(x)\,dx$, is

"integral"

"(area above x-axis) − (area below x-axis)"

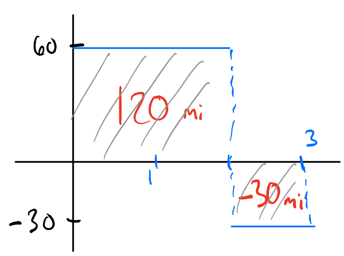


Why we need signed area (an example):

Consider a road trip:

$0 \le t \le 2$ driving <u>away from</u> home at 60 mph

$2 \le t \le 3$ driving <u>towards</u> home at 30 mph



How far from home (net dist.) are you?

<u>Ans 1</u>: Signed area under curve = 120 mi − 30 mi = 90 mi

<u>Ans 2</u>: GPS readings: $x(3) - x(0) = 90$ mi $- 0$ mi $= 90$ mi.

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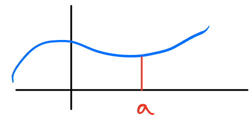
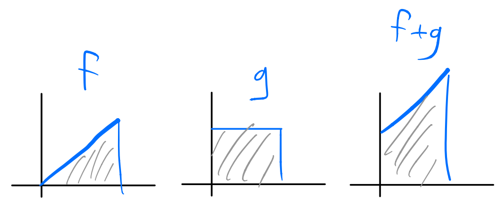
Properties of signed area:

① $\int_a^a f(x)\,dx = 0$

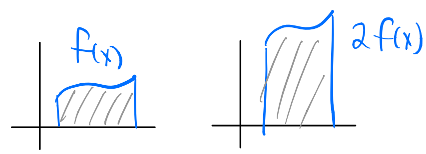


② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

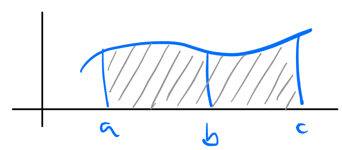
③ $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$



④ $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$



⑤ $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

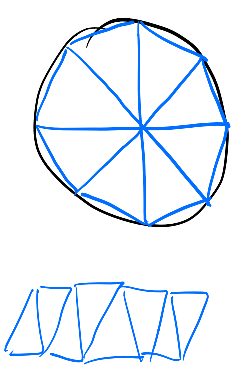


Recall that we had a limit definition of the derivative:

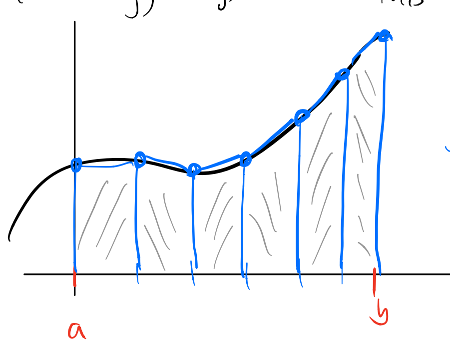
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \text{slope of tangent line.}$$

Now, we'll need a limit definition for (signed) area under the curve.

This is motivated by Archimedes limit definition of the area of a circle.



One (of many) ways to do this

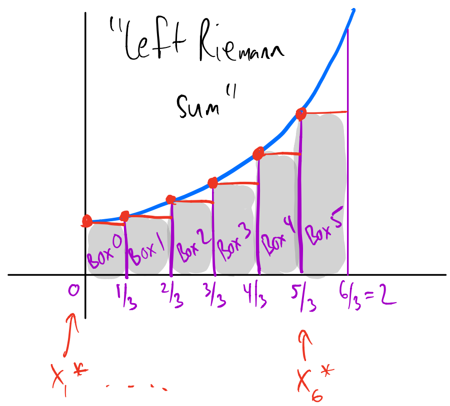


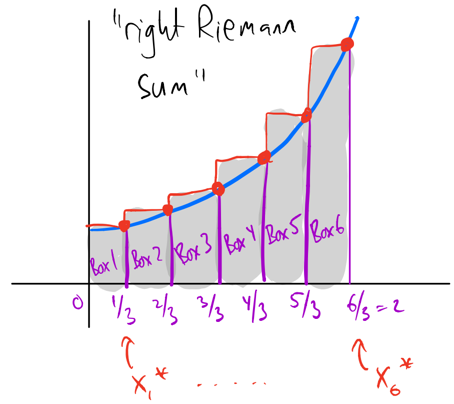
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\text{area of } \square)$$

$= ???$

Riemann sums "approximate area under the curve"

Example: Approximate the area under $f(x) = x^2+1$ from $a=0$ to $b=2$.





Here, we are subdividing the interval $[a,b] = [0,2]$ into $n=6$ equal parts, each one having width $\Delta x = 2/6 = 1/3$.

Left Riemann sum: Area $= f(0)\cdot\Delta x + f\left(\frac{1}{3}\right)\Delta x + f\left(\frac{2}{3}\right)\Delta x + f\left(\frac{3}{3}\right)\Delta x + f\left(\frac{4}{3}\right)\Delta x + f\left(\frac{5}{3}\right)\Delta x$

right Riemann sum: Area $= f\left(\frac{1}{3}\right)\Delta x + f\left(\frac{2}{3}\right)\Delta x + f\left(\frac{3}{3}\right)\Delta x + f\left(\frac{4}{3}\right)\Delta x + f\left(\frac{5}{3}\right)\Delta x + f\left(\frac{6}{3}\right)\Delta x$

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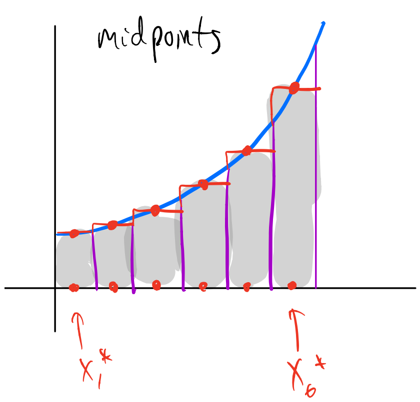
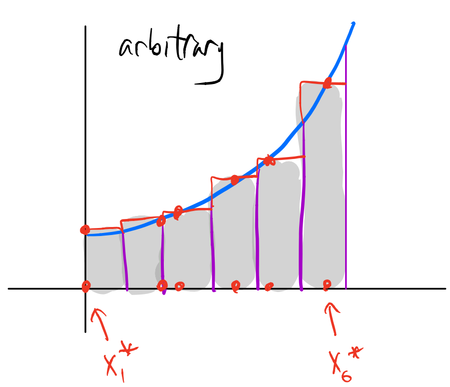
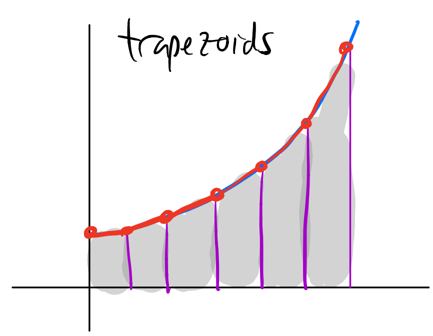
Alternatively, one can approximate the area using the midpoint of each box, or any other point, or with trapezoids.







Regardless of which Riemann sum we choose,

$$\text{Area} = \lim_{\Delta x \to 0} \left(\sum_{i=1}^{n} \text{area of } i^{th} \text{ box} \right)$$

$$= \lim_{\Delta x \to 0} \left(\underset{\displaystyle \downarrow \atop \displaystyle \int_a^b}{\sum_{i=1}^{n}} \ \underset{\displaystyle \downarrow \atop \displaystyle f(x)}{f(x_i^*)} \cdot \underset{\displaystyle \downarrow \atop \displaystyle dx}{\Delta x} \right) := \int_a^b f(x)\, dx$$

Let's compute this explicitly for $f(x) = x^2+1$, i.e., $\int_0^2 (x^2+1)\, dx$.

First, we'll review "sigma notation":

$$\sum_{k=1}^{6} k = 1+2+3+4+5+6 = \sum_{i=1}^{6} i$$

"dummy variable"

<u>Properties</u>: $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ "break apart sums"

$\displaystyle\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$ "pull out constants"

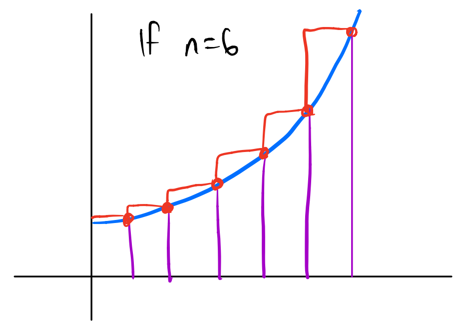
<u>Identities</u>: $\displaystyle\sum_{k=1}^{n} k = 1+2+3+\cdots+(n-1)+n = \frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k^2 = 1+4+9+\cdots+(n-1)^2+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1+8+27+\cdots+(n-1)^3+n^3 = \frac{n^2(n+1)^2}{4}$$

Riemann sum example: Compute $\int_0^2 (x^2+1)\,dx$. Let $\Delta x = \frac{2-0}{n} = \frac{2}{n}$.

Subintervals $[0, 2/n], [2/n, 4/n], \ldots, [2(n-1)/n, 2]$



Right endpoints: $2/n, 4/n, \ldots, 2i/n, \ldots, 2$

$$\text{Area} = \sum_{i=1}^{n} f(x_i^*)\cdot\Delta x$$

$$= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n}$$

$$= \sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+1\right]\cdot\frac{2}{n} = \sum_{i=1}^{n}\left(\frac{8i^2}{n^3}+\frac{2}{n}\right)$$

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$$= \sum_{i=1}^{n}\frac{8i^2}{n^3} + \sum_{i=1}^{n}\frac{2}{n}$$

$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1$$

$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{2}{n}\cdot[n]$$

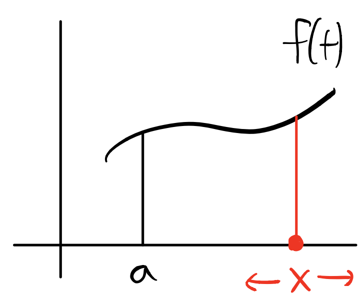
Now, take $\lim_{n\to\infty}$:
$$\lim_{n\to\infty}\frac{8}{6}\cdot\frac{n(n+1)(2n+1)}{n^3} + \lim_{n\to\infty} 2$$

$$= \frac{4}{3}\cdot 2 + 2 = \boxed{\frac{14}{3}}$$

Area function: Fix $f(x)$ and a real number, a.

Define $A(x) = \int_a^x f(t)\,dt$

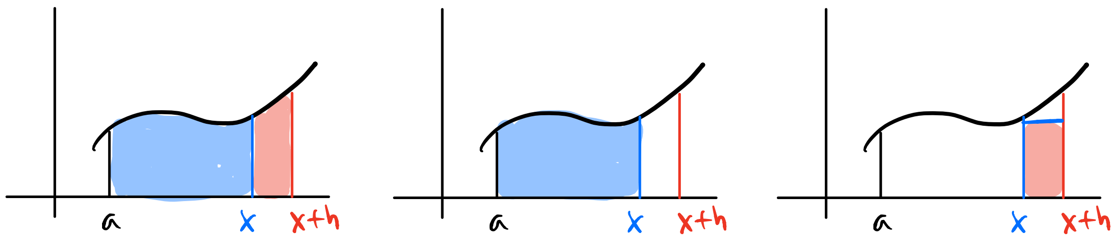
$=$ area under the curve from a to x



Clearly, $A(a) = \int_a^a f(t)\,dt = 0.$

Remark:

$$A(x+h) \quad - \quad A(x) \quad \approx \quad f(x)\cdot h$$



Note that $A(x+h) - A(x) \approx f(x)\cdot h$

$$\Rightarrow \frac{A(x+h)-A(x)}{h} \approx f(x)$$

Take $\lim_{h\to 0}$ of both sides: $\boxed{A'(x) := \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x)}$

Big idea: If $A(x)$ is the area function of $f(x)$, then $\frac{d}{dx}\big(A(x)\big) = f(x)$. In other words:

"the derivative & area functions are inverse operations"

$$\begin{tikzcd} f(x) \arrow[r, bend left, "d/dx"] & f'(x) \arrow[l, bend left, "\text{area under curve}"] \end{tikzcd} \qquad \begin{tikzcd} x(t) \arrow[r, bend left, "d/dt"] & v(t) \arrow[l, bend left, "\text{area under curve}"] \end{tikzcd}$$

This is the Fundamental theorem of calculus, Part 1

$\boxed{\text{If } f \text{ is continuous on } [a,b] \text{ and differentiable on } (a,b), \text{ then } \frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$

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We say that $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Antiderivatives are not unique!

Antiderivatives of $f(x) = 2x$ include x^2, x^2+1, $x^2+2, \ldots$

Fact: If $F(x)$, $G(x)$ are antiderivatives of $f(x)$, then

$F(x) - G(x) = C$, for some constant.

Why: $(F-G)' = F' - G' = C - C = 0 \Rightarrow F - G = C$.

Now, consider a function $f(x)$.

We know $A(x)$ is an antiderivative (by FTC 1)

Let $F(x)$ be any other antiderivative.

Then $F(x) = A(x) + C$

Recall: $A(a) = 0$

$$\Rightarrow F(b) - F(a) = (A(b) + C) - (A(a) + C)$$

$$= A(b) = \int_a^b f(x)\,dx$$

This is the Fundamental theorem of calculus, Part 2

$\boxed{\text{If } f \text{ is continuous on } [a,b] \text{ and } F \text{ is any antiderivative of } f, \text{ then } \int_a^b f(x)\,dx = F(b) - F(a)}$