

Part 7: Computing integrals

Math 1060

Fall 2020

Oct 21-30



Wed 10/21

□

Notation: If $F'(x) = f(x)$, i.e., if $F(x)$ is an antiderivative of $f(x)$, then we write $\int f(x) dx = F(x) + C$. We call this an indefinite integral.

In contrast, $\int_a^b f(x) dx$ is a definite integral

Practice with antiderivatives (indefinite integrals)

- $\int 1 dx = x + C$
- $\int x dx = \frac{1}{2}x^2 + C$
- $\int x^2 dx = \frac{1}{3}x^3 + C$

General formula: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $n \neq -1$

Fri 10/23

$$\bullet \int \sqrt{x} dx = \int x^{1/2} dx = \frac{x^{3/2}}{3/2} + C = \frac{2}{3} x^{3/2} + C = \frac{2}{3} \sqrt{x^3} + C$$

$$\bullet \int \frac{1}{x} dx = \int x^{-1} dx = \ln x + C \quad [\text{Note: } \frac{x^0}{0} \text{ is undefined}]$$

$$\bullet \int e^x dx = e^x + C$$

$$\bullet \int e^{kx} = \frac{1}{k} e^{kx} + C$$

$$\bullet \int \sin x dx = -\cos x + C$$

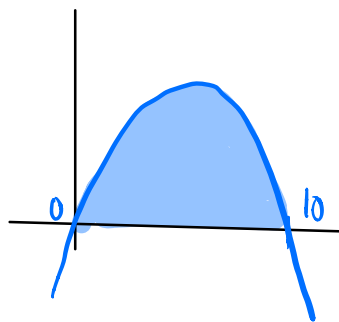
$$\bullet \int \sin kx dx = -\frac{1}{k} \cos kx + C$$

$$\bullet \int \cos x dx = \sin x + C$$

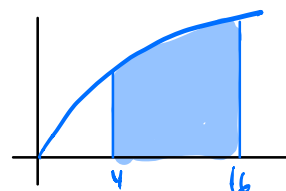
$$\bullet \int \cos kx dx = \frac{1}{k} \sin kx + C$$

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Practice with definite integrals

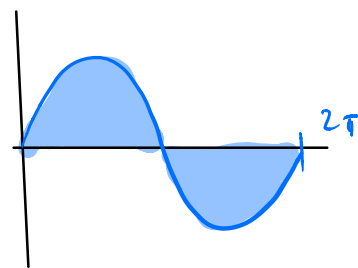
$$\begin{aligned} \bullet \int_0^{10} 60x - 6x^2 &= \left(\frac{60x^2}{2} - \frac{6x^3}{3} \right) \Big|_0^{10} \\ &= (30x^2 - 2x^3) \Big|_0^{10} \\ &= (30 \cdot 10^2 - 2 \cdot 10^3) - (30 \cdot 0^2 - 2 \cdot 0^3) \\ &= 3000 - 2000 = \boxed{1000} \end{aligned}$$



$$\begin{aligned} \bullet \int_4^{16} 3\sqrt{x} \, dx &= \int_4^{16} 3x^{1/2} \, dx = \frac{3x^{3/2}}{3/2} \Big|_4^{16} \\ &= 2x^{3/2} \Big|_4^{16} = 2\sqrt{x^3} \Big|_4^{16} = 2\sqrt{16^3} - 2\sqrt{4^3} = 2 \cdot 64 - 2 \cdot 8 = \boxed{112} \end{aligned}$$



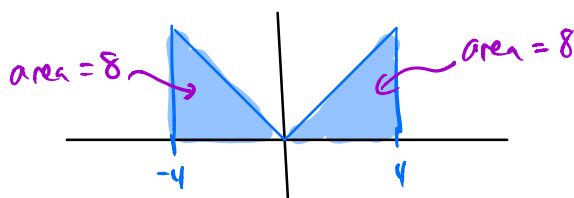
$$\begin{aligned} \bullet \int_0^{2\pi} 3 \sin x \, dx &= -3 \cos x \Big|_0^{2\pi} \\ &= (-3 \cos 2\pi) - (-3 \cos 0) \\ &= -3 \cdot 1 + 3 \cdot 1 = \boxed{0} \end{aligned}$$



Mon 10/26

$$\bullet \int_{-4}^4 |x| \, dx = 16 \text{ (see graph)}$$

$$\begin{aligned} \text{OR} &= \int_{-4}^0 -x \, dx + \int_0^4 x \, dx \\ &= -\frac{x^2}{2} \Big|_{-4}^0 + \frac{x^2}{2} \Big|_0^4 \\ &= -(0 - 8) + (8 - 0) = \boxed{16} \end{aligned}$$



Application: Average value of a function.

Motivating example: through 8 games,

Clemson	8 wins
Oklahoma	7 wins
USC	3 wins

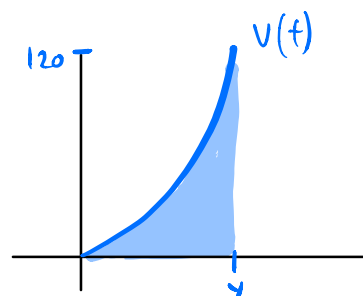
The average # wins = $\frac{8+7+3}{3} = \frac{18}{3} = 6$ wins.

Now, consider a continuous quantity, $f(x)$.

The average value of $f(x)$ on $a \leq x \leq b$ is $\frac{1}{b-a} \int_a^b f(x) dx$.

Ex 1: Suppose the velocity of a car is given by $v(t) = 30t^2$ for $0 \leq t \leq 2$. Find its average velocity.

$$\begin{aligned} \text{Ans: } \frac{1}{2-0} \int_0^2 30t^2 dt &= \frac{1}{2} \left. \frac{30t^3}{3} \right|_0^2 \\ &= 5t^3 \Big|_0^2 = 5 \cdot 2^3 - 5 \cdot 0 = 40 \end{aligned}$$



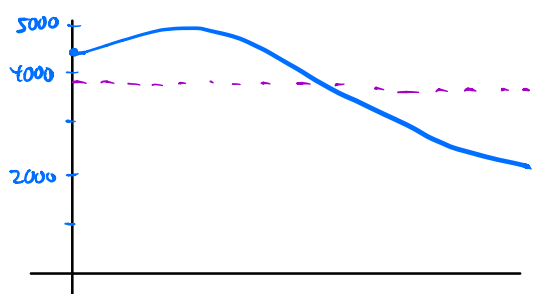
Ex 2: Suppose the elevation on a 5-mile hike is

$$f(x) = 60x^3 - 650x^2 + 1200x + 4500 \text{ ft.}$$

Find the average elevation.

Ans:

$$\begin{aligned} \frac{1}{5-0} \int_0^5 (60x^3 - 650x^2 + 1200x + 4500) dx \\ = \frac{1}{5} \left(15x^4 - \frac{650x^3}{3} + 600x^2 + 4500x \right) \Big|_0^5 \end{aligned}$$



$$\textcircled{4} = \frac{1}{5} (15 \cdot 5^4 - \frac{650}{3} \cdot 5^3 + 600 \cdot 5^2 + 4500 \cdot 5) \approx \boxed{3958.3 \text{ ft}}$$

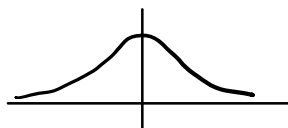
Derivatives vs. integrals

Derivatives have formulas: product, quotient, chain rules.

So we can always take the derivative of, e.g., $e^{\left[\sin\left(\frac{x^2}{x+1}\right) \cdot \cos x\right]}$.

Integrals have no such formula.

For example, $\int e^{-x^2} dx$ has no "closed form."



Computing integrals is more of an "art".

But there are some rules & "tricks." We'll learn one now.

Wed 10/28

u-substitution "like the chain rule in reverse"

Recall the chain rule: $\frac{d}{dx} (F(g(x))) = f(g(x)) \cdot g'(x)$ [where $F' = f$]

Now integrate both sides:

$$F(g(x)) + C = \int \underbrace{f(g(x))}_{du} \cdot \underbrace{g'(x) dx}_{du}$$

Now let $u = g(x)$

$$\Rightarrow \frac{du}{dx} = g'(x)$$

$$\Rightarrow du = g'(x) dx$$

$$\boxed{F(u) + C = \int f(u) du}$$

Example

$$\textcircled{1} \int 2(2x+1)^{30} dx \quad \text{let } u = 2x+1 \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2 dx$$

$$= \int u^{30} du = \frac{1}{31} u^{31} + C = \frac{1}{31} (2x+1)^{31} + C$$

$$\begin{aligned} \textcircled{2} \int 3e^{10x} dx & \quad \text{let } u=10x \Rightarrow \frac{du}{dx}=10 \Rightarrow du=10 dx \\ & = \int e^u \cdot \frac{3}{10} du \quad \Rightarrow \frac{3}{10} du = 3 dx \\ & = \frac{3}{10} e^u + C = \frac{3}{10} e^{10x} + C. \end{aligned}$$

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$$\begin{aligned} \textcircled{3} \int x^4(x^5+6)^9 dx & \quad \text{let } u=x^5+6, \frac{du}{dx}=5x^4 \Rightarrow du=5x^4 dx \\ & = \int u^9 \cdot \frac{1}{5} du \quad \Rightarrow \frac{1}{5} du = x^4 dx \\ & = \frac{1}{5} \cdot \frac{u^{10}}{10} + C = \frac{1}{50} u^{10} + C = \frac{1}{50} (x^5+6)^{10} + C \end{aligned}$$

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$$\begin{aligned} \textcircled{4} \int (\cos x)^3 \sin x dx & \quad \text{let } u=\cos x \Rightarrow du=-\sin x dx \\ & = \int u^3 (-du) = -\frac{u^4}{4} + C = \frac{(\cos x)^4}{4} + C \end{aligned}$$

$$\begin{aligned} \textcircled{5} \int \frac{x dx}{\sqrt{1+x^2}} & \quad \text{let } u=1+x^2 \Rightarrow du=2x dx \Rightarrow \frac{1}{2} du = x dx \\ & = \int \frac{1}{2} u^{-1/2} du = \frac{1}{2} \frac{u^{1/2}}{1/2} + C = u^{1/2} + C = \sqrt{1+x^2} + C \end{aligned}$$

$$\begin{aligned} \textcircled{6} \int 3x^2 \sqrt{x^3+1} dx & \quad \text{let } u=x^3+1 \Rightarrow du=3x^2 dx \\ & = \int u^{1/2} du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3} \sqrt{u^3} + C = \frac{2}{3} \sqrt{(x^3+1)^3} + C \end{aligned}$$

$$\begin{aligned} \textcircled{7} \int \frac{\sin t}{2-\cos t} dt & \quad \text{let } u=2-\cos t \Rightarrow du=\sin t dt \\ & = \int -\frac{1}{u} du = -\ln|u| + C = -\ln(2-\cos t) + C \end{aligned}$$