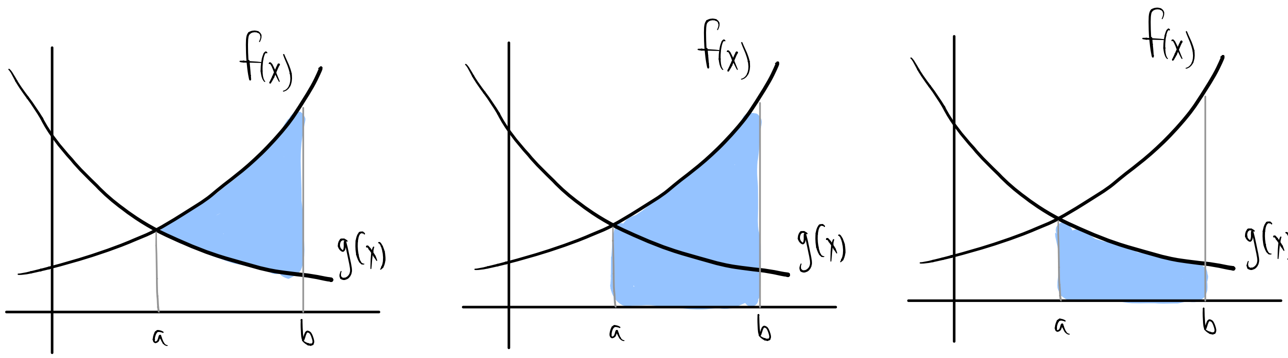
Part 8: Applying integrals

Math 1060
Fall 2020
Oct 30 – 11/1

In this section, we'll learn how to compute various volumes of solids using integrals. Then we'll apply these techniques to several architectural structures.

Area between curves



$$\int_a^b [f(x) - g(x)]\, dx \quad = \quad \int_a^b f(x)\, dx \quad - \quad \int_a^b g(x)\, dx$$

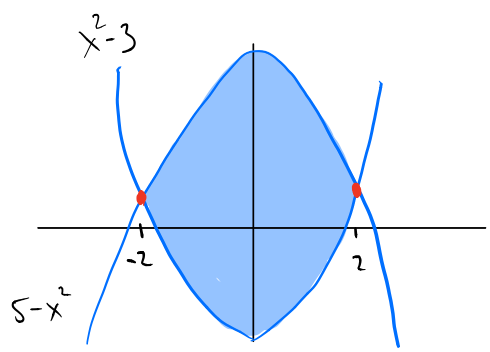
This even works if the region is below or straddles the x-axis

Example 1: Find the area between the curves of $f(x) = 5 - x^2$ & $g(x) = x^2 - 3$.

First, find points of intersection:

$$5 - x^2 = x^2 - 3$$

$$8 = 2x^2 \Rightarrow x \pm 2.$$



$$\text{Area} = \int_{-2}^{2} (5 - x^2) - (x^2 - 3)\, dx$$

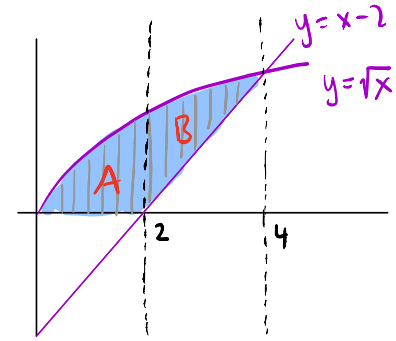
$$= \int_{-2}^{2} (8 - 2x^2)\, dx = 8x - \frac{2}{3}x^3 \Big|_{-2}^{2} = \left(16 - \frac{16}{3}\right) - \left(-16 - \frac{16}{3}\right) = 32 - \frac{32}{3} = \boxed{\frac{64}{3}}$$

Example 2: Find the area between the curves $y = \sqrt{x}$, $y = x - 2$, and the x-axis.

$\boxed{2}$

Method 1: Integrate w.r.t. x.

Note that we need to break this into two integrals.



Ⓐ $\displaystyle\int_0^2 \sqrt{x} - 0 \, dx = \int_0^2 x^{1/2} \, dx = \left.\frac{x^{3/2}}{3/2}\right|_0^2 = \left.\frac{2}{3}\sqrt{x^3}\right|_0^2 = \boxed{\frac{2}{3}\sqrt{8}}$

Ⓑ $\displaystyle\int_2^4 \sqrt{x} - (x-2) \, dx = \int_2^4 x^{1/2} - x + 2 \, dx = \left.\left(\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right)\right|_2^4$

$= \left(\frac{2}{3}\sqrt{4^3} - \frac{16}{2} + 8\right) - \left(\frac{2}{3}\sqrt{2^3} - \frac{4}{2} + 2\right)$

$= \left(\frac{2}{3}\cdot 8 - \cancel{8} + \cancel{8}\right) - \left(\frac{2}{3}\sqrt{8} - 2 + 4\right) = \frac{16}{3} - \frac{2}{3}\sqrt{8} + 2 = \boxed{\frac{10}{3} - \frac{2}{3}\sqrt{8}}$

Total area $=$ Ⓐ $+$ Ⓑ $= \boxed{\boxed{\frac{10}{3}}}$

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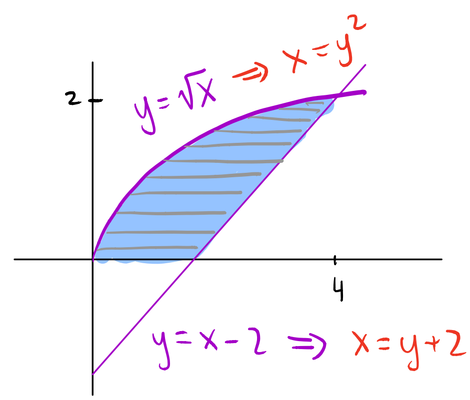
Method 2: Integrate w.r.t. y.



Area $= \displaystyle\int_0^2 (y+2) - y^2 \, dy$

$= \left.\left(\frac{y^2}{2} + 2y - \frac{y^3}{3}\right)\right|_0^2$

$= \left[\left(2+4-\frac{8}{3}\right) - (0+0-0)\right]$

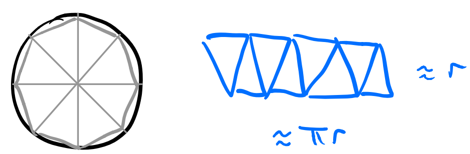
$= \boxed{\frac{10}{3}}$ (much easier!)

Volumes by slicing

We'll now learn to derive classic formulas for volumes such as

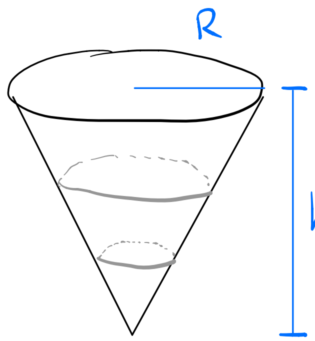
$$\text{vol}(\text{cone}) = \frac{1}{3}\pi r^2 h \quad \text{and} \quad \text{vol}(\text{sphere}) = \frac{4}{3}\pi r^3.$$

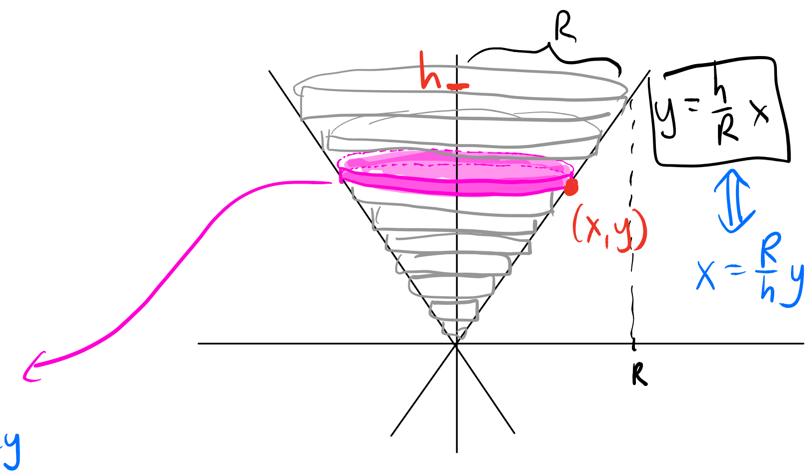
The method is in some sense, a 3D-version of how Archimedes computed the area of a circle.

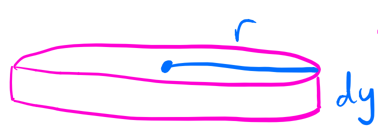


Volume of a cone

Idea: Slice the cone into layers, like a wedding cake. Each layer is ≈ cylinder.







$$\text{Area} = \pi r^2 h, \quad \text{radius } r = x\text{-value} = \frac{R}{h}y$$

$$= \pi\left(\frac{R}{h}y\right)^2 dy = \boxed{\frac{\pi R^2}{h^2} y^2\, dy}$$

$$\text{Vol}(\nabla) = \int_0^h \text{vol}(\text{disk}) = \int_0^h \frac{\pi R^2}{h^2} y^2\, dy$$

$$= \frac{\pi R^2}{h^2}\int_0^h y^2\, dy = \frac{\pi R^2}{h^2}\frac{y^3}{3}\Big|_0^h = \frac{\pi R^2}{h^2}\left(\frac{h^3}{3} - \frac{0^3}{3}\right) = \boxed{\frac{1}{3}\pi R^2 h} \checkmark$$

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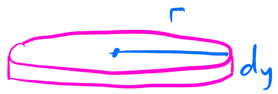
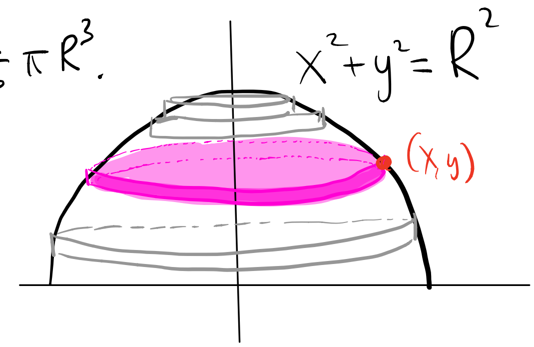
Volume of a hemisphere $\frac{1}{2}\cdot\left(\frac{4}{3}\pi R^3\right) = \frac{2}{3}\pi R^3.$





radius $r = x\text{-value} = \sqrt{R^2-y^2}$

$$\mathrm{Vol}(\;) = \pi r^2 h = \pi\left(\sqrt{R^2-y^2}\right)^2 dy$$
$$= \boxed{\pi\left(R^2-y^2\right) dy}$$

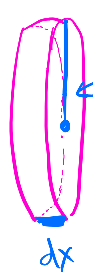
$$x^2+y^2=R^2$$
$$\Rightarrow x = \pm\sqrt{R^2-y^2}$$

$$\text{Vol of hemisphere} = \int_0^R \mathrm{Vol}(\;)$$

$$= \int_0^R \pi\left(R^2-y^2\right) dy = \int_0^R \left(\pi R^2 - \pi y^2\right) dy$$

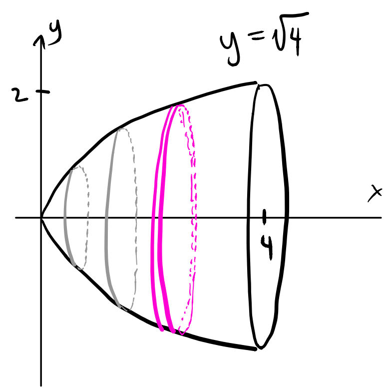
$$= \left(\pi R^2 y - \pi\frac{y^3}{3}\right)\Big|_0^R = \pi R^3 - \pi\frac{R^3}{3} = \boxed{\frac{2}{3}\pi R^3} \checkmark$$

Example 3: Consider the solid formed by revolving the curve $y=\sqrt{x}$ around the x-axis from $x=0$ to $x=4$. Find its volume.



radius = "y-value": $y=\sqrt{x}$

$$\mathrm{Vol}(\;) = \pi r^2 h$$
$$= \pi\left(\sqrt{x}\right)^2 dx$$
$$= \pi x\, dx$$



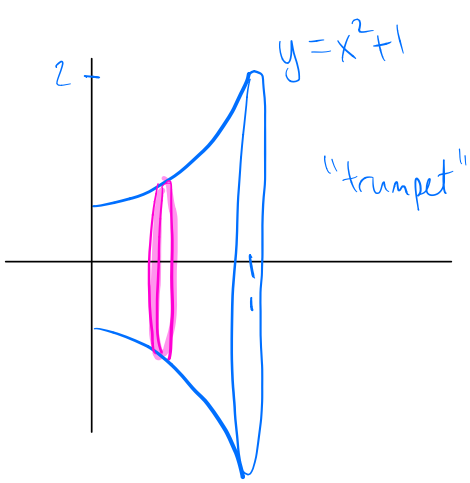
$$\mathrm{Vol}(\;) = \int_0^4 \mathrm{Vol}(\;) = \int_0^4 \pi x\, dx = \pi\frac{x^2}{2}\Big|_0^4 = \boxed{8\pi}$$

These are called **solids of revolution**.

The method we've been doing is called the **disk method**.

We can do other shapes:

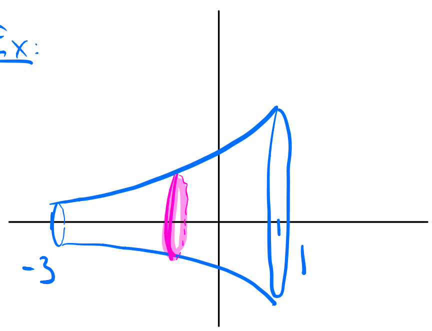
Ex:



$$\text{Vol}\left(\quad\right) = \int_1^2 \text{Vol}\left(\quad\right)$$

$$= \int_1^2 \pi (x^2+1)^2 \, dx$$

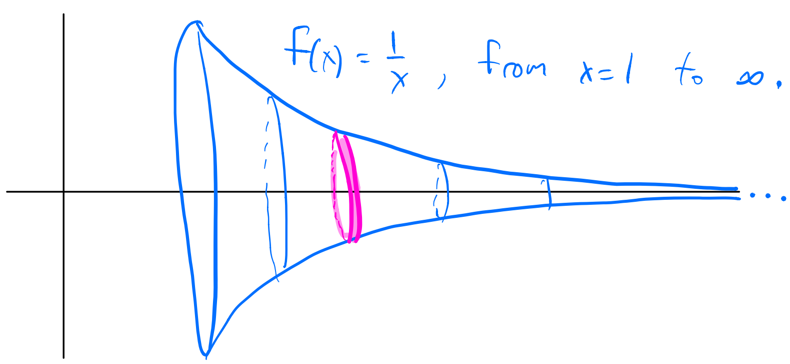
Ex:



$$\text{Vol}\left(\quad\right) = \int_{-3}^{1} \text{Vol}\left(\quad\right)$$

$$= \int_1^2 \pi (e^x)^2 \, dx = \int_1^2 \pi e^{2x} \, dx$$

Ex: "Gabriel's horn"



First, we need to see what are called "**improper integrals**", i.e, integrating over an asymptote, or where a limit is ∞.

Big idea: "treat ∞ as an ordinary number."

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Examples: • $\int_1^\infty \frac{1}{x}\,dx = \ln x \Big|_1^\infty = \ln \infty - \ln 1 = \infty - 0 = \boxed{\infty}$

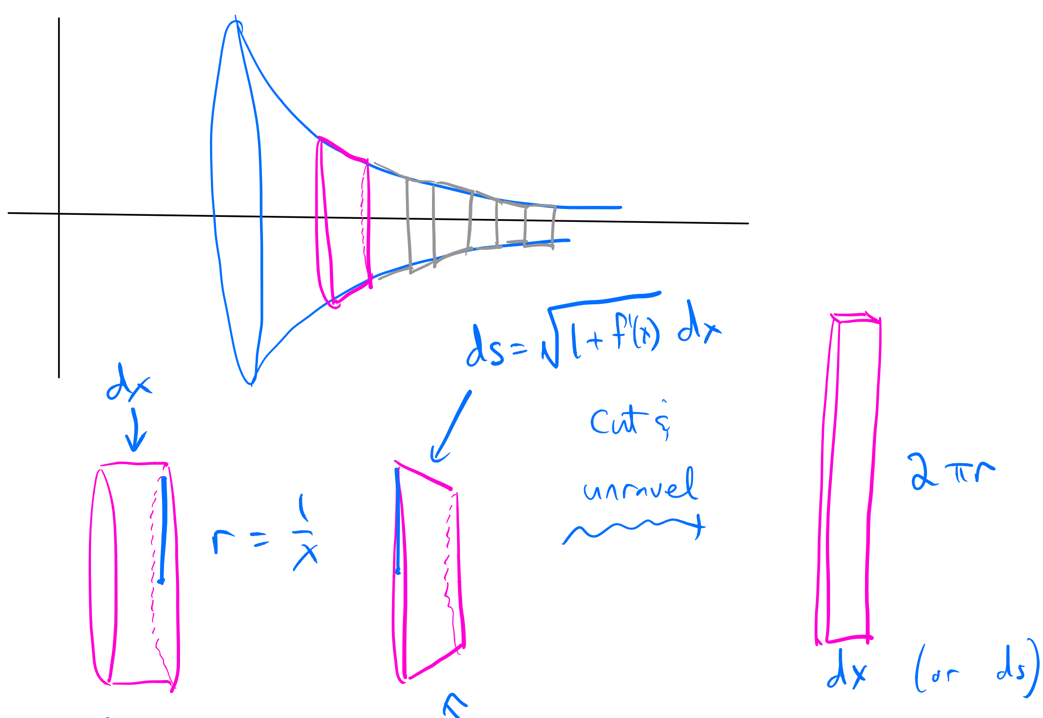
[Technically, $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_1^b = \lim_{b\to\infty} \ln b$]

• $\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty} - \left(-\frac{1}{1}\right) = 0 + 1 = \boxed{1}$

Back to Gabriel's horn:

$$\mathrm{Vol}(\quad) = \int_1^\infty \mathrm{Vol}(\quad) = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi \boxed{\int_1^\infty \frac{1}{x^2}\,dx} = \boxed{\pi}$$

Just for fun, we can compute the surface area.



Surface area $= 2\pi r\,dx$
$= 2\pi \frac{1}{x}\,dx$ $\leq$ Surface area $= 2\pi r\,ds$
$= 2\pi r \sqrt{1+f'(x)}\,dx$

Thus, surface area $= \int_1^\infty SA(\quad)$

$$\geq \int_1^\infty SA(\quad) = \int_1^\infty 2\pi \frac{r}{x}\,dx$$

$$= 2\pi r(\ln x)\Big|_1^\infty = 2\pi r(\ln \infty - \ln 1) = \boxed{\infty}$$

Thus, Gabriel's horn has finite volume, but infinite surface area.

"We can fill it with paint, but not paint the whole surface"

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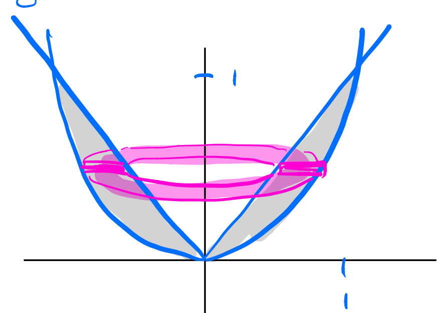
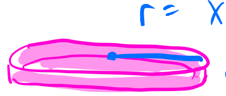
The following technique is sometimes called "volumes by washers"

Ex: Compute the volume of the region b/w $y = x^2$ and $y = x$, rotated around the y-axis.



$r =$ "x-value"



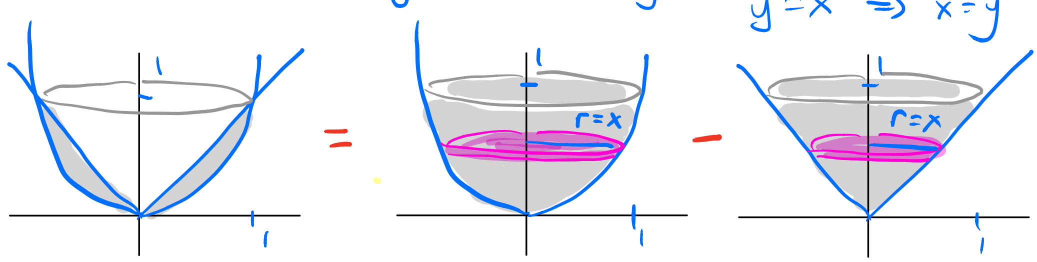
dx

vol $= \pi r^2\,dx$

$y = x^2 \Rightarrow x = \sqrt{y}$

$y = x \Rightarrow x = y$

Big idea:

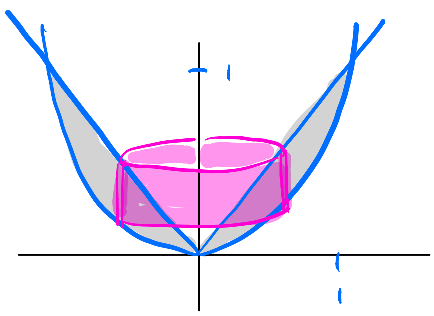


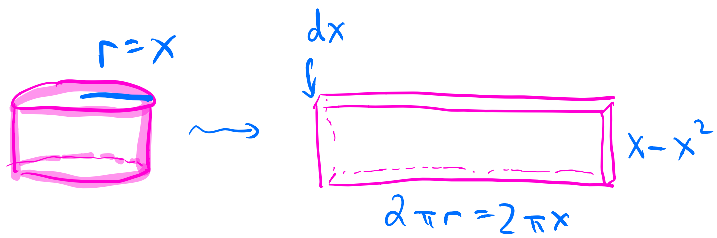
$$\int_0^1 (\pi y - \pi y^2)\,dy = \int_0^1 \pi(\sqrt{y})^2\,dx - \int_0^1 \pi y^2\,dx$$

$$= \left(\frac{\pi y^2}{2} - \frac{\pi y^3}{3}\right)\Big|_0^1 = \left(\frac{\pi}{2} - \frac{\pi}{3}\right) - (0 - 0) = \boxed{\frac{\pi}{6}}$$

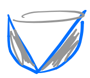
Another way to find the volume of the previous solid is the "**shell method**"

Ex:





$$\mathrm{Vol} = 2\pi r h \cdot dx = 2\pi x(x - x^2)\,dx$$

$\mathrm{Vol}\Big($  $\Big) = \int_0^1 \mathrm{Vol}\Big($  $\Big)$

$$= \int_0^1 2\pi x(x-x^2)\,dx = \int (2\pi x^2 - 2\pi x^3)\,dx$$

$$= \left(\frac{2\pi x^3}{3} - \frac{2\pi x^4}{4}\right)\Bigg|_0^1 = 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1$$

$$= 2\pi\left[\left(\frac{1}{3} - \frac{1}{4}\right) - (0-0)\right] = \boxed{\frac{\pi}{6}}$$

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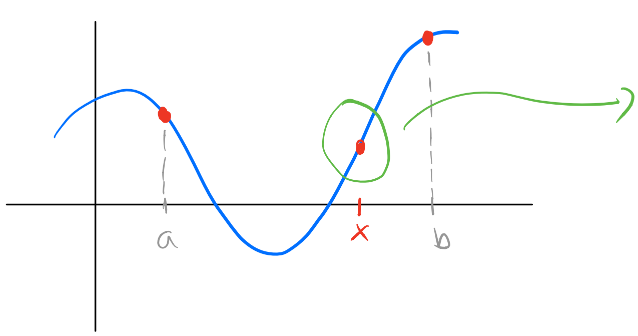
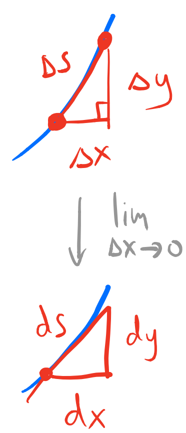
Arc length

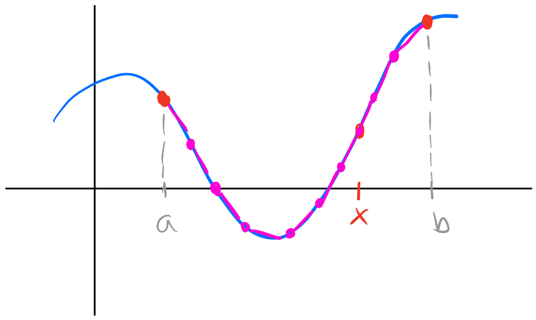
Goal: Find the length of a curve $y = f(x)$ from $x = a$ to $x = b$.





Δs = arc length

ds = "infintesmal arc length"



$$\text{Arc length} = \lim_{\Delta s \to 0} \sum_{i=1}^{n} \Delta s = \int_a^b ds$$

need formula

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 + (dy)^2} \cdot \frac{dx}{dx} = \sqrt{\left[(dx)^2 + (dy)^2\right]\left(\frac{dx}{dx}\right)^2}$$

$$= \sqrt{\left[\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\right](dx)^2} = \sqrt{1 + (f'(x))^2}\, dx$$

Thus, $\boxed{\text{arc length} = \int_a^b \sqrt{1+(f'(x))^2}\, dx}$

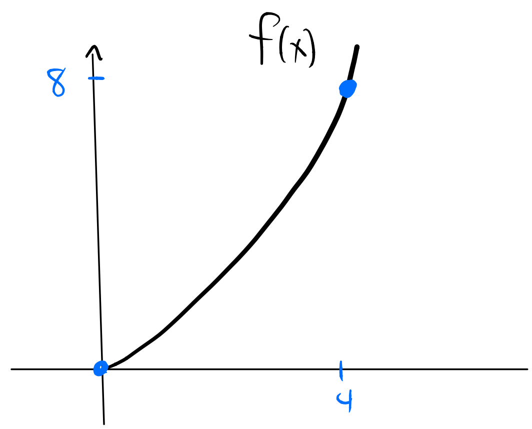
Examples:

1. Find the arc length of $y = \sqrt{x^3} = x^{3/2}$ from $x=0$ to $x=4$

$$\int_0^4 \sqrt{1 + \left(\frac{d}{dx} x^{3/2}\right)^2}\, dx$$

$$= \int_0^4 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2}$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4} x}\, dx$$



$\boxed{10}$ let $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}\,dx \Rightarrow dx = \frac{4}{9}\,du$

$$= \int_{x=0}^{x=4} \sqrt{u} \cdot \frac{4}{9}\,du = \frac{4}{9}\int_{x=0}^{x=4} u^{1/2}\,du = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]\Big|_{x=0}^{x=4}$$

$$= \frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\Big|_0^4 = \frac{8}{27}\cdot 10^{3/2} - \frac{8}{27}\cdot 1^{3/2} = \frac{8}{27}\left(\sqrt{1000}-1\right)$$

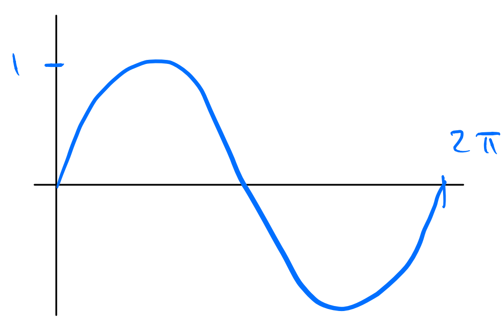
$$\approx 9.073.$$

Remark: Often, one of these integrals $\int \sqrt{1+(f'(x))^2}\,dx$ ends up being too complicated (e.g., no closed form sol'n, or we haven't learned the method). For these, Wolfram Alpha is helpful.

Ex 2: Compute the arc length of one full cycle of $\sin x$

$$\int_0^{2\pi} \sqrt{1+\left(\frac{d}{dx}\sin x\right)^2}\,dx$$

$$= \int_0^{2\pi} \sqrt{1+(\cos x)^2}\,dx$$



≈ 7.640 (by Wolfram Alpha)