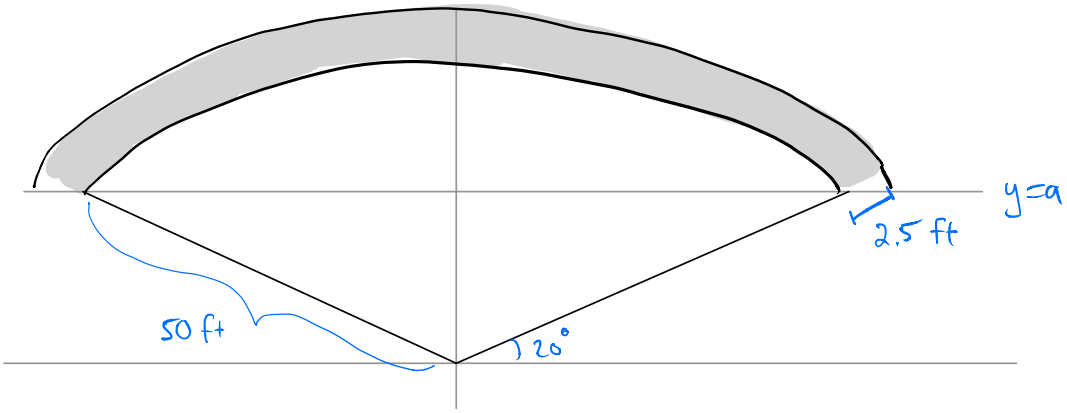
Part 9: Calculus of domes

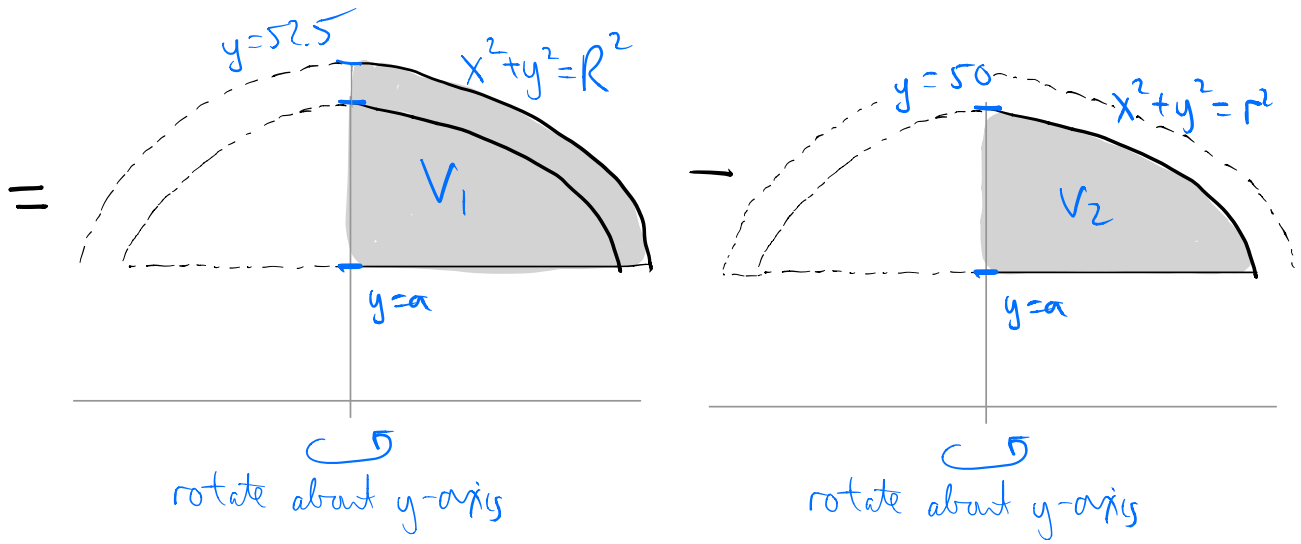
Math 1060
Fall 2020
Nov 13 - 20

Fri 11/13

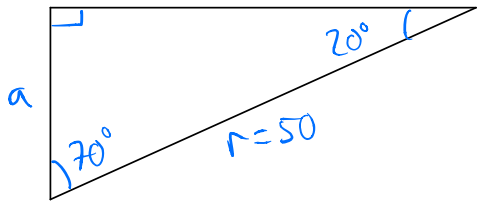
1. Hagia Sophia : See PPT slides for history and overview

Our goal is to calculate the weight of the dome.





Find a:



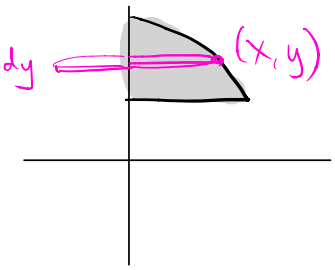
$\cos 70^\circ = \frac{a}{50}$

$\Rightarrow a = 50 \cos 70 \approx 17.1$

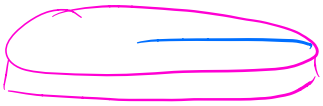
So $R = 52.5$, $r = 50$, $a = 17.1$

[2]

Find V_1:



radius = x-value $= \sqrt{R^2 - y^2}$



$$V_1 = \int_a^R \pi (\text{radius})^2 \, dy = \int_a^R \pi \sqrt{R^2 - y^2}^2 \, dy$$

$$= \int_a^R \pi (R^2 - y^2) \, dy = \pi \left(R^2 y - \frac{y^3}{3} \right) \Big|_{a=17.1}^{R=52.5}$$

$$= \pi \left(R^3 - \frac{R^3}{3} \right) - \pi \left(R^2 a - \frac{a^3}{3} \right)$$

$$= \pi \left[\frac{2}{3} R^3 - R^2 a + \frac{1}{3} a^3 \right] \approx 160233 \text{ ft}^3$$

$$V_2 = \int_a^r \pi (\text{radius})^2 \, dy = \int_a^r \pi \sqrt{r^2 - y^2}^2 \, dy$$ ← same as V_1 but replace R with r

$$\ldots = \pi \left[\frac{2}{3} r^3 - r^2 a + \frac{1}{3} a^3 \right] \approx 132733 \text{ ft}^3$$

So $V_1 - V_2 = 27500 \text{ ft}^3$

The concrete in the dome weighs $\approx 110 \text{ lb/ft}^3$

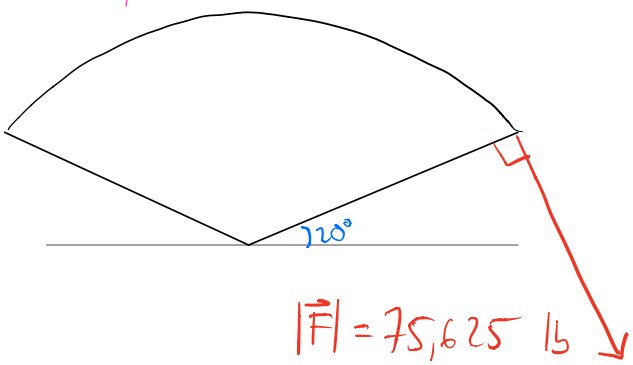
$\Rightarrow$ total weight is $(27500 \text{ ft}^3)(110 \text{ lb/ft}^3)$

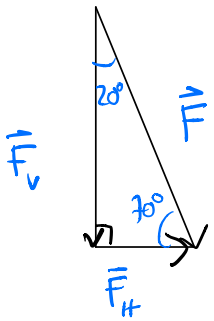
$= \boxed{3{,}025{,}000 \text{ lb}}$.

There are 40 supporting ribs $\Rightarrow \frac{3,025,000}{40} = 75,625$ lbs/rib.

Let's calculate the outward force at each buttress

Mon 11/16





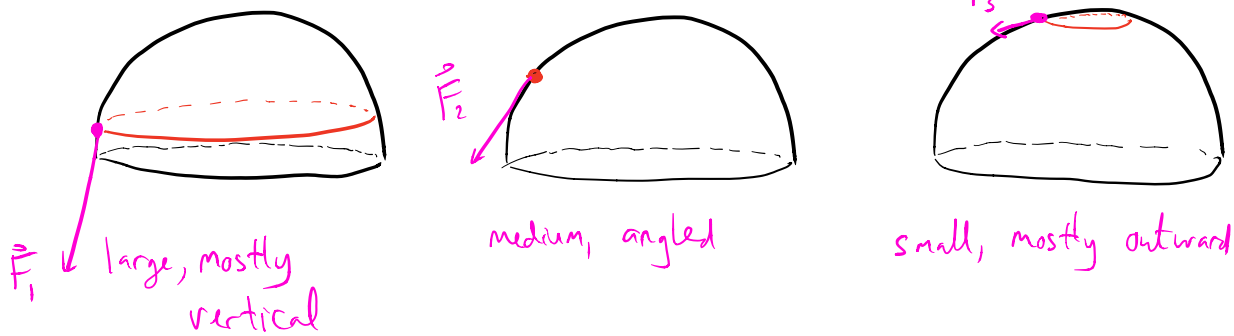
Basic trig: $|\vec{F}_V| = |\vec{F}| \cdot \cos 20 \approx 71,064$ lb

$|\vec{F}_h| = |\vec{F}| \cos 70 \approx 25,865$ lb

Def: • Vertical cross-sections of a dome are called meridians

• Horizontal cross-sections of a dome are called hoops

Compare the force (magnitude & direction) of the dome on a hoop, at various heights.



(4)

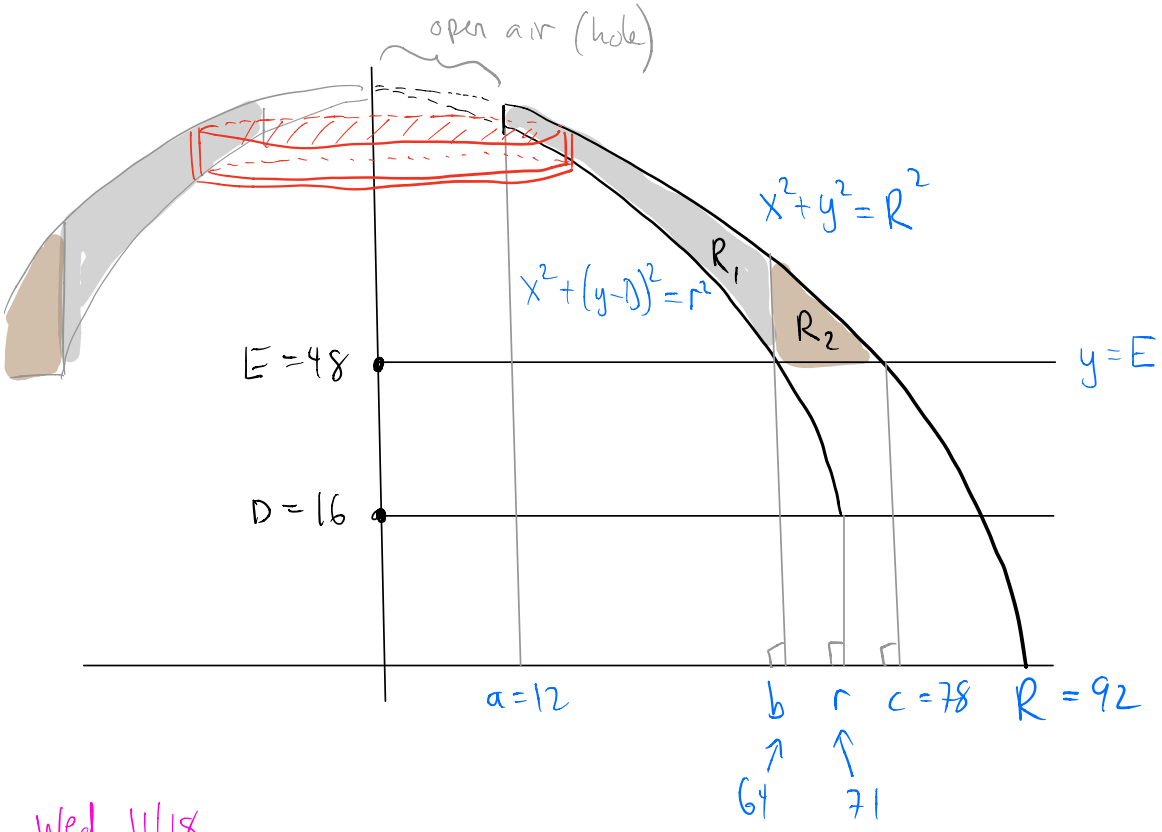
The hoop stress is the outward force.

★ First principle of structural architecture: Unless the shell can resist the "hoop stress," it will expand along the hoops & cracks will develop along meridians.

(2) Roman Pantheon: See PPT slides for history & overview.

Two key differences b/w Pantheon & Hagia Sophia:

1. The inner & outer shells are spheres with different centers
2. The brick & mortar was mixed with pumice at the top to reduce hoop stress.



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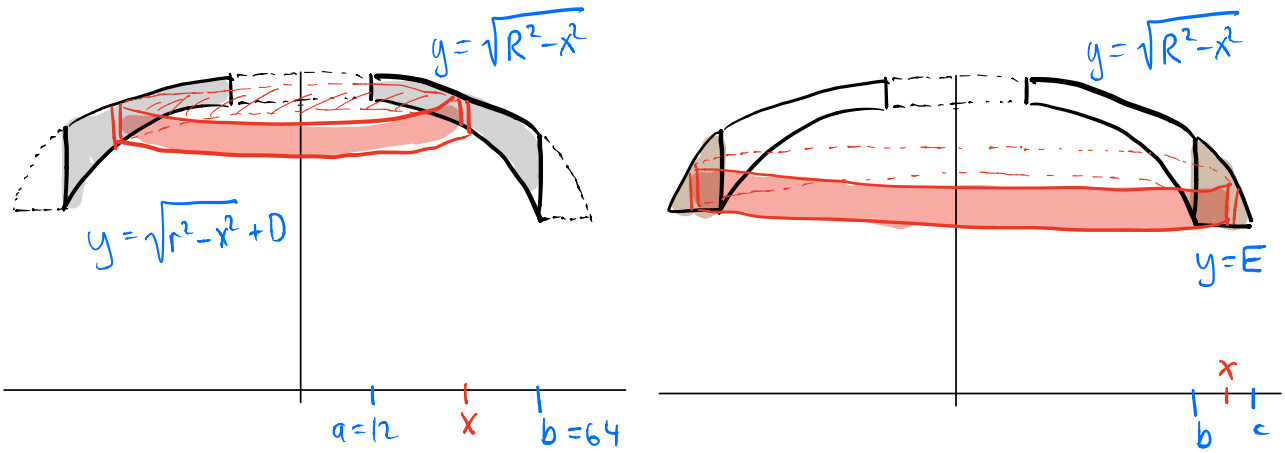
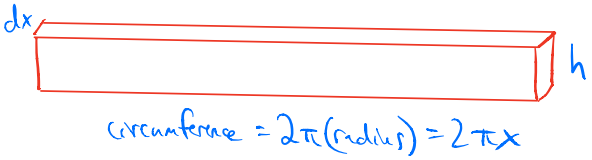
We'll use the shell method



Recall: Vol of a shell



$$\text{Vol } I = \int_a^b 2\pi x \left[\sqrt{R^2-x^2} - \left(\sqrt{r^2-x^2}+D\right)\right] dx$$

Let $u = R^2 - x^2$

$\Rightarrow du = -2x\,dx$

$$= \int_a^b 2\pi x \sqrt{R^2-x^2}\,dx - \int_a^b 2\pi x \sqrt{r^2-x^2}\,dx - \int_a^b 2\pi x D\,dx$$

$$= \int_{x=a}^{x=b} -\pi\sqrt{u}\,du + \int_{x=a}^{x=b} \pi\sqrt{u}\,du - \int_a^b (2\pi D)\,x\,dx$$

$$= -\frac{2}{3}\pi u^{3/2}\Big|_{x=a}^{x=b} + \frac{2}{3}\pi u^{3/2}\Big|_{x=a}^{x=b} - \pi D x^2\Big|_a^b$$

$$= -\frac{2}{3}\pi\left(R^2-x^2\right)^{3/2}\Big|_a^b + \frac{2}{3}\pi\left(r^2-x^2\right)^{3/2}\Big|_a^b - \pi D\left(b^2-a^2\right)$$

$$= -656{,}875 + 984{,}818 - 198{,}694 = \boxed{129{,}299\ \text{ft}^3}$$

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$$\text{Vol } 2 = \int_b^c 2\pi x \left[\sqrt{R^2-x^2} - E\right] dx$$

$$= \int_b^c 2\pi x \sqrt{R^2-x^2}\, dx - \int_b^c 2\pi E x \, dx$$

$$= -\frac{2}{3}\pi\left(R^2-x^2\right)^{3/2}\Big|_b^c - \pi E x^2\Big|_b^c$$

$$= 361442 - 299783$$

$$= \boxed{61659 \text{ ft}^3}$$

So, the total volume of the dome is the sum:

$$129,299 + 61659 = \boxed{190958 \text{ ft}^3}$$