Part 10: Calculus of arches

Math 1060

Fall 2020

11/23 – 12/3

The shape of an ideal arch (see slides on Canvas)

What is an ideal arch?

Three conditions:

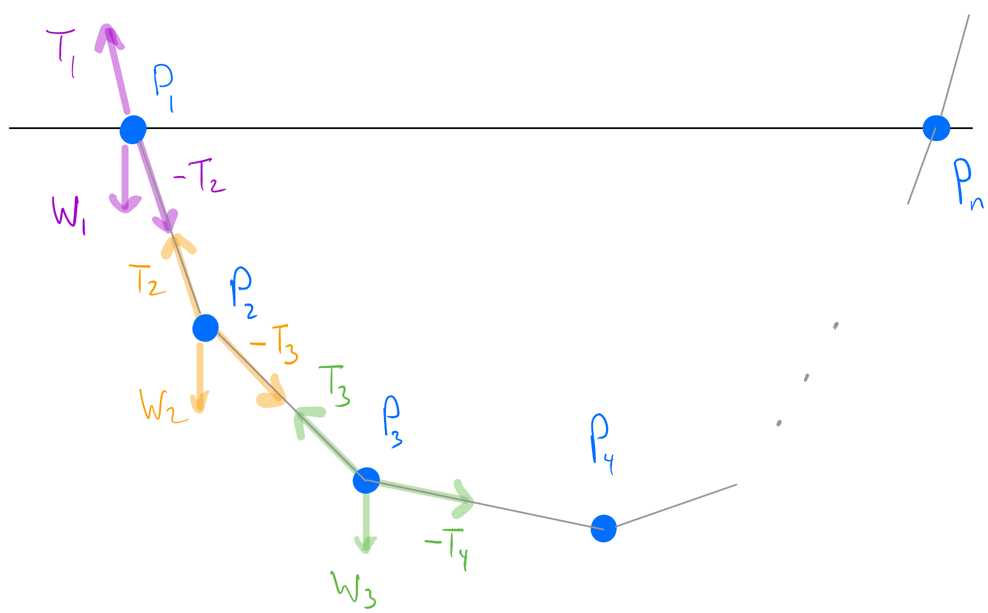
① The only load on the arch is its weight

② The only external support is at its base

③ Gravitational forces on the arch are balanced perfectly by its reaction to the compression that these forces generate.

Another answer, due to Robert Hooke (1671): "hanging chain, upside-down"

Imagine a weightless fishing line that has weights strung along it.



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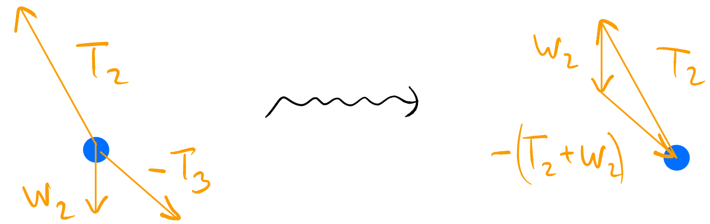
☆ Forces must balance if the string is at rest.

At each weight:



$\Rightarrow T_3 = T_2 + W_2$

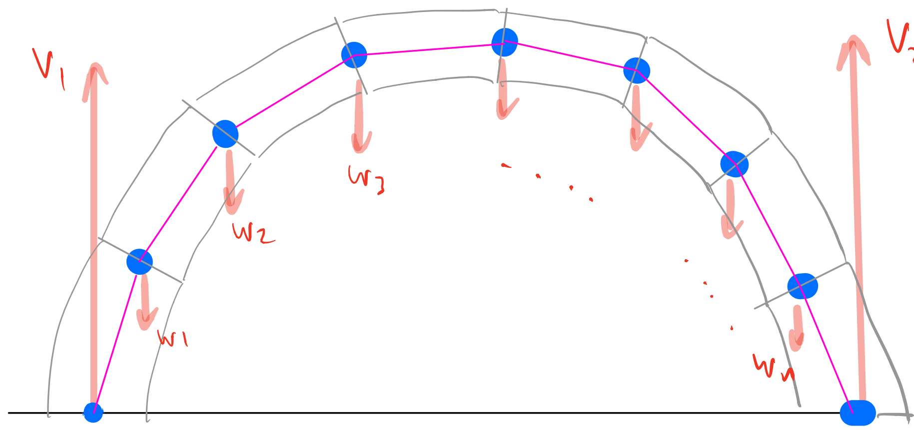
Thus, we get a system of equations that must hold:

$$\begin{cases} T_1 + W_1 = T_2 \\ T_2 + W_2 = T_3 \\ T_3 + W_3 = T_4 \\ \quad\vdots \\ T_n + W_n = T_{n+1} \end{cases}$$

... which is <u>not</u> fun to solve.

As usual, the "Calculus version" of this problem is less messy (later).

If we turn a chain upside-down, we get an arch.

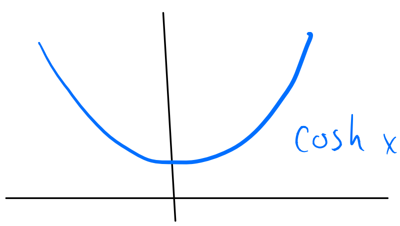


Stability $\Rightarrow$ $W_1 + W_2 + \dots + W_n = V_1 + V_2$

Safe theorem (Heyman, 1966): "As long as there is any polygonal path inside the arch such that the forces balance, then the arch is stable.

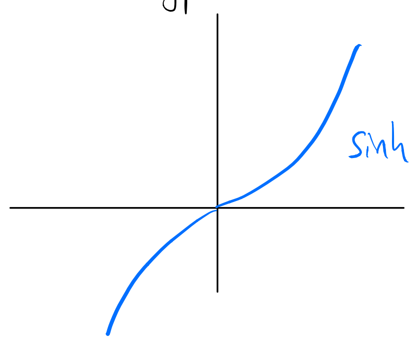
It turns out that the shape of an ideal arch, or hanging chain (i.e, take $\lim_{n\to\infty}$ in the previous examples) is a hyperbolic cosine function, $A\cosh(Bx) = A\left(\frac{e^{Bx}+e^{-Bx}}{2}\right)$, $AB=1$.

Def: Hyperbolic cosine



$\cosh x = \frac{e^{x}+e^{-x}}{2}$

Hyperbolic sine



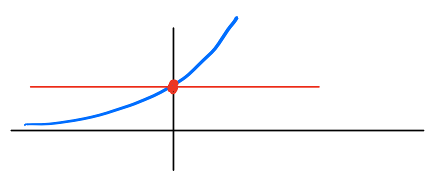
$\sinh x = \frac{e^{x}-e^{-x}}{2}$

Let's explore this further...

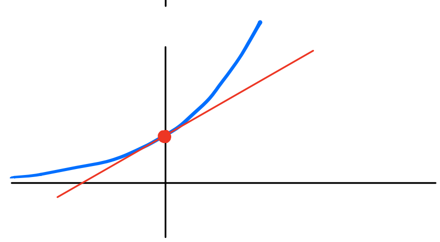
Polygonal approximations of functions. (Topic from the end of Calculus 2)

Motivating example: Let's approximate $f(x) = e^x$ at $x=0$.

- 0^{th} order approximation: $T_0(x) = 1$ (y-intercept)



- 1^{st} order approximation: $T_1(x) = 1 + x$ (We've done a lot of these...)



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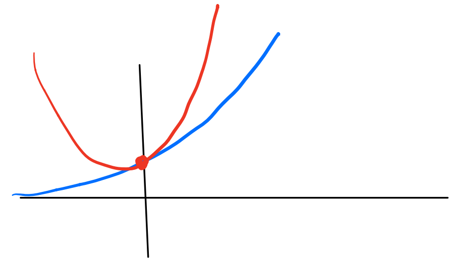
- 2^{nd} order approximation:

$$T_2(x) = 1 + x + \frac{1}{2}x^2$$



And so on...

- n^{th} order approximation: $T_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

Taking the limit as $n \to \infty$ yields an infinite series:

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$$e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

☆ How to find a_n?

Note that
$$T_n(0) = a_0 = e^0 = 1$$
$$T_n'(0) = a_1 = e^0 = 1 \quad \text{since } \frac{d}{dx} e^x = e^x$$
$$T_n''(0) = 2a_2 = e^0 = 1 \quad \text{since } \frac{d^2}{dx^2} e^x = e^x$$
$$T_n'''(0) = 6a_3 = e^0 = 1$$
$$\vdots$$
$$T_n^{(k)}(0) = k!\, a_k = e^0 = 1$$

Thus, the coefficient of x^k is $\boxed{a_k = \frac{1}{k!}}$

That is, $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \ldots$

Therefore: $e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \ldots$

Add these! $\frac{e^x + e^{-x}}{2} = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 = \cosh x$

Subtract these! $\frac{e^x - e^{-x}}{2} = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 = \sinh x$

$\cosh 0 = \cos 0 = 1$
$\sinh 0 = \sin 0 = 0$

Remarks

- $e^x = \cosh x + \sinh x$
- $\frac{d}{dx}(\sinh x) = \cosh x$, $\frac{d}{dx}(\cosh x) = \sinh x$
- If $i = \sqrt{-1}$, then $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, ...

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} +/- \dots$$

$$e^{-ix} = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \frac{x^8}{8!} +/- \dots$$

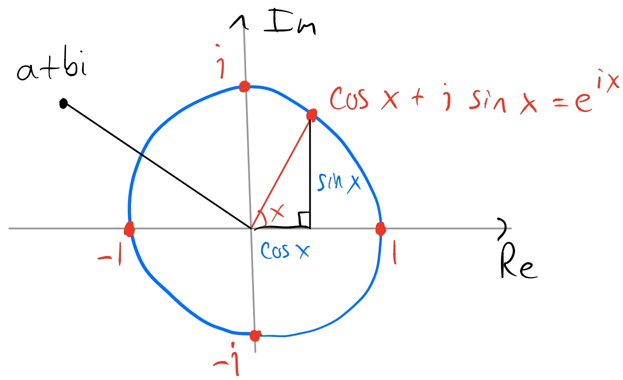
$$\frac{e^{ix}+e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} +/- \dots = \cos x$$

$$\frac{e^{ix}-e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +/- \dots = \sin x$$

Remarks:

- $e^{ix} = \cos x + i \sin x$
- $e^{i\pi} = \cos \pi + i \sin \pi$

$\Rightarrow \boxed{e^{i\pi} = -1}$!!!



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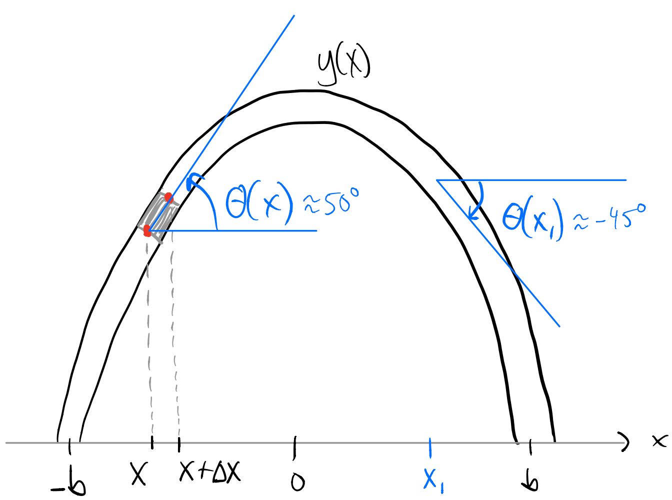

Consider an ideal arch.

Let $C(x)$ = compression force at x

$\theta(x)$ = angle at x

$y(x)$ = graph of the function of the arch.

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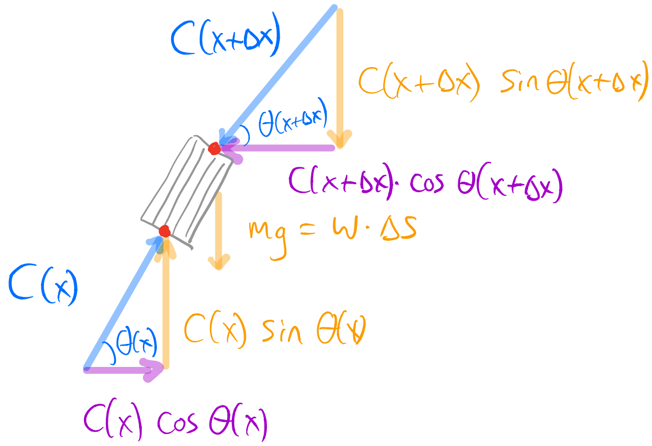


☆ Need to balance horizontal & vertical forces.

Balance horizontal forces

$$C(x+\Delta x)\cdot \cos\theta(x+\Delta x) - C(x)\cos\theta(x) \approx 0$$

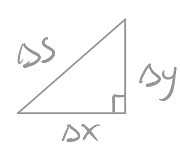
$$\lim_{\Delta x\to 0} \frac{C(x+\Delta x)\cdot\cos\theta(x+\Delta x) - C(x)\cos\theta(x)}{\Delta x} \approx 0$$ take the limit of both sides

$$\int_{-b}^{x} \frac{d}{dx} C(x)\cos\theta(x)\, dx = \int_{-b}^{x} 0 \, dx$$ integrate both sides

(☆) $C(x)\cos\theta(x) = C_0$ $\qquad C_0$ = unknown constant

Remark: If we plug in $x=0$, we get $C(0)\underbrace{\cos\theta(0)}_{=1} = C_0$

Thus $C_0 = C(0)$ = compression force at top of arch.



Balance vertical forces

$$C(x+\Delta x)\sin\theta(x+\Delta x) - C(x)\sin\theta(x) \approx -w\cdot\boxed{\Delta S} \longrightarrow \sqrt{(\Delta x)^2+(\Delta y)^2}\cdot\sqrt{\frac{(\Delta x)^2}{(\Delta x)^2}}$$

$$= \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\cdot \Delta x$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{as } \Delta x \to 0$$

$$\lim_{\Delta x \to 0} \frac{C(x+\Delta x)\sin\theta(x+\Delta x) - C(x)\sin\theta(x)}{\Delta x} \approx -w\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$ take limit of both sides

$$\int_{-b}^{x} \frac{d}{dx} C(x)\sin\theta(x)\,dx = -w\int_{-b}^{x}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$(\star\star)\quad C(x)\sin\theta(x) = -w\underbrace{\int_{-b}^{x}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}_{\text{length of arch from } -b \text{ to } x}$$

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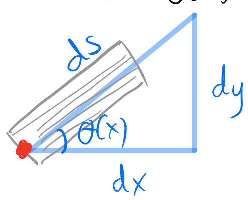
Now, what do we do with $(\star)$ and $(\star\star)$?

Consider $\tan\theta(x)$: there are two ways to compute it.

① 

$$\tan\theta(x) = \frac{dy}{dx}$$

② $$\frac{(\star\star)}{(\star)} = \frac{C(x)\sin\theta(x)}{C(x)\cos\theta(x)}$$

$$= -\frac{w}{C_0}\int_{-b}^{x}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Thus, the function $y(x)$ that describes the arch satisfies

$$\boxed{\frac{dy}{dx} = -\frac{w}{C_0}\int_{-b}^{x}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

This is an example of a differential equation: an equation that defines $y(x)$ implicitly, but not explicitly.

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$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[-\frac{w}{C_0}\int_{-b}^{x}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\right]$$

$$(*) \quad \boxed{\frac{d^2y}{dx^2} = -\frac{w}{C_0}\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$ by Fund. Thm. Calc.

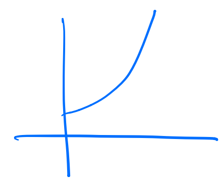
Aside: Differential equations come up in science & engineering.

Ex: "Rate of change of an investment is proportional to its value"

$$y' \quad = \quad k \quad y$$

e.g., 5% interest rate: $y' = 0.05\,y$

This has sol'n $y(t) = Ce^{0.05t}$ "exponential growth"

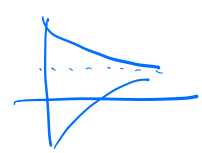


Another example:

"Rate of change of temp. of coffee is proportional to difference b/w room temp"

$$T' \quad = \quad k \quad (72^\circ - T)$$

$T' = k(72 - T)$ has sol'n $T(t) = 72 + Ce^{-kt}$



Back to solving $(*)$.

Claim: $\boxed{y(x) = -\frac{C_0}{w}\cosh\left(\frac{w}{C_0}x\right) + D}$ works.

Recall: $\cosh kx = \dfrac{e^{kx}+e^{-kx}}{2}$, $\quad \sinh kx = \dfrac{e^{kx}-e^{-kx}}{2}$

$$(\cosh kx)^2 = \tfrac{1}{4}\left(e^{2kx} + 2 + e^{-2kx}\right), \quad (\sinh kx)^2 = \tfrac{1}{4}\left(e^{2kx} - 2 + e^{-2kx}\right)$$

$$(\cosh kx)^2 - (\sinh kx)^2 = 1$$

Also $\frac{d}{dx} \cosh kx = \sinh kx$ and $\frac{d}{dx} \sinh kx = \cosh kx$

So $\frac{dy}{dx} = -\sinh\left(\frac{w}{C_0} x\right)$

$$\frac{d^2y}{dx^2} = -\frac{w}{C_0} \cosh\left(\frac{w}{C_0} x\right) \leftarrow \text{LHS (need to show this is } = \text{RHS).}$$

RHS: $-\frac{w}{C_0}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = -\frac{w}{C_0}\sqrt{1+\left(\sinh \frac{w}{C_0}x\right)^2} = -\frac{w}{C_0}\sqrt{\left(\cos \frac{w}{C_0}x\right)^2}$

$$= -\frac{w}{C_0}\cosh\left(\frac{w}{C_0}x\right) \quad \checkmark$$

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Next: If h is the max height of an arch, then

$$y(0) = -\frac{w}{C_0}\underbrace{\cosh 0}_{=1} + D = -\frac{w}{C_0} + D = h \Rightarrow \boxed{D = h + \frac{w}{C_0}}$$

Thus the final solution is $\boxed{y(x) = -\frac{C_0}{w}\cosh\left(\frac{w}{C_0}x\right) + \left(h + \frac{w}{C_0}\right)}$

In general, the shape of an ideal arch is $y(x) = A\cosh\left(\frac{x}{A}\right) + D$.

Now, back to the St. Louis Arch... (see Canvas slides).